Oscillons in Driven Granular Systems

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Abstract

In a vertically-oscillated shallow layer of beads, long-lived localized structures called 'oscillons' have been observed. These structures oscillate at half of the driving frequency, and may be either in phase or directly out of phase with the driving force at the peak of a cycle. Oscillons interact with eachother and have been observed form linear polymer-like structures and also 2D lattices. Oscillons have also been observed in a variety of other systems, including the acoustics of stellar envelopes, networks of parametrically driven damped oscillators , and the Swift-Hohenberg model.

1 Introduction

Systems composed of hard granules show a large range of behaviors shared with other systems which make them a rich area of study. Depending on the density of grains, and the presence of confining forces, they can behave as a gas, liquid, or solid. They exhibit a phenomenon known as 'shear banding' [7]: when a shear force is applied, grains in a layer near the surface display far more than grains in the bulk (i.e. most of the stresses do not communicate past this surface layer). When a ball bearing is dropped into a pile of sufficiently fine grains, a jet is formed which has been observed to clump (despite the absence of surface-tension between the grains) [6].

When a thin layer of liquid (water, for example) is vibrated, various patterns of standing waves are observed to form on the surface. These are known as Faraday waves, and exhibit striped and square-grid phases [8]. The same thing happens in a vibrated thin layer of grains. However, additional behaviors are observed in the granular case which are not observed in the case of a pure liquid.

For certain ranges of frequency and driving amplitude, localized surface structures are observed. These structures are peaks and dips that oscillate at half of the driving frequency, so that at the peak of a driving cycle, they alternate between 'up' and 'down' states. Correspondingly, two of these localized structures (oscillons) can be either in phase or a half-cycle out of phase with eachother. These oscillons are observed to interact and form bound structures.

It seems possible that oscillons are topological defects of some sort, caused by the freezing out of some degree of freedom in a frequency and amplitude dependant transition. However, oscillons have been observed to decay spontaneously (though only after very long times), and can be created by locally perturbing the oscillating granular layer. Also, 'up' and 'down' oscillons do not annihilate, but rather form bound structures which can either be stationary or may propagate in one direction [2]. This brings up the question of what is responsible for the long lifetime of these localized structures.

Granular systems are not the only systems in which this sort of structure is observed. There are mathematical similarities between the governing equations for oscillons in granular materials and (among others) the Swift-Hohenberg equation[2], and the equations modelling acoustic waves in the outer envelopes of stars [1]. In [2], it is proposed that oscillons are an intermediary structure that occur in any system that has the squares-stripes transitions so long as that transition has certain properties (subcritical).

These systems can be studied experimentally (it is probably easiest to study the granular case as it deals with easily-observed macroscopic objects), by simulation (either by simulating the dynamics of hard spheres directly or by using one of the equations that has been proposed to model the evolution of the 'surface' height), or by analytic techniques in non-equilibrium pattern formation, such as linear stability analysis.

2 Oscillon-Forming Systems

2.1 Granular Layers

Oscillons have been observed experimentally in granular layers which were vibrated at the appropriate (range of) amplitude and frequency. In [4] oscillons were observed at a frequency of 26 Hz. The frequencies for which oscillons occured were found to depend on the diameter of the spheres used. The amplitude (described by a dimensionless factor $\Gamma = 4\pi^2 A f^2 g^{-1}$) corresponded to $\Gamma = 2.45$.

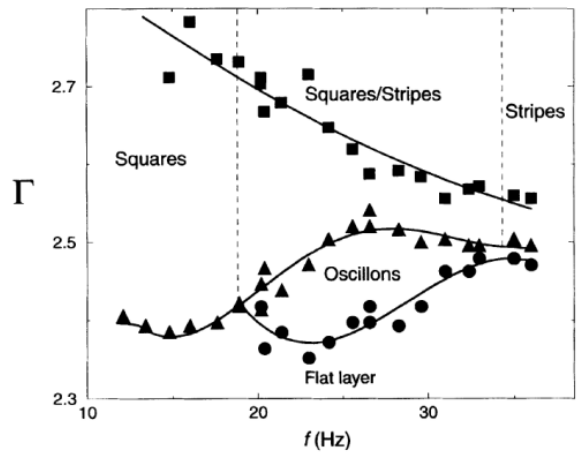

Figure 1: The phase diagram for a vertically oscillated granular layer. The figure is from [4].

Whether or not oscillons appeared was determined to depend on the depth of the layer. The minimum layer depth for which oscillons were observed was 13 particles. On average, the width of oscillons was observed to be 30 particles. The region in which oscillons were found is between the square phase and the homogeneous phase. They found that oscillons would not spontaneously occur from the homogeneous phase, but that they would have to be generated by starting with a square pattern and decreasing the amplitude through the transition. Oscillons would then be formed near defects in the square pattern. The experimentally determined phase diagram for this system is shown in Fig (1).

In order to simulate oscillons in a granular system directly, one must efficiently handle the interactions between many objects. In 2D, oscillons can be observed in systems containing on the order of 7×10^3 hard discs. The 3D case requires about 2×10^5 spheres. In order to handle this many interacting objects, it is necessary for the computational complexity to be O(N). This can be done by segmenting the computational region into boxes such that only spheres in adjacent boxes can interact. That way, for each sphere, there is a maximum number of checks which is independant of the total number of spheres. Another issue with simulating hard sphere interactions is preventing overlap. If one sphere is displaced in order to remove its overlap with a second, that displacement can make it then overlap with a large number of neighbors. So fully relaxing the stresses in a pile of spheres may take an unreasonably long time. One solution is to permit a small number of overlap, and use a 'soft core' potential, so that spheres can slip past eachother more easily in a jammed situation. Another partial solution is to use an adaptive timestep, such that short time steps are taken only when the overlap error is large.

The benefit of using a hard-sphere simulation to model oscillons is that one can keep track of the trajectories of individual spheres, and, compared to other techniques which approximate the behavior of the surface, a direct simulation is much closer to the actual experiment. However, direct simulation is slow, and it is difficult to know when you have actually obtained the behavior you are looking for (if the oscillon is very small, it may be obscured by 'gassified' particles that are ejected from the bulk).

In [5], a model based on conservation of mass and a non-linear response to compression was proposed. There are two fields in the model: the local density of grains ρ and a complex order parameter ψ. The order parameter represents oscillations in the surface at half the driving frequency (the driving frequency oscillations and integrated out and contribute only a driving term). The equations of motion are [5]:

$$\partial_t \psi = \gamma \psi^* - (1 - i\omega)\psi + (1 + ib)\nabla^2 \psi - |\psi|^2 \psi - \rho \psi$$

$$\partial_t \rho = \alpha \nabla \cdot (\rho \nabla |\psi|^2) + \beta \nabla^2 \rho$$

In the first equation, the $\gamma\psi^*$ term drives the oscillations. It breaks the symmetry under changes of the phase of ψ, so corresponds to synchronization/desynchronization of the subharmonic oscillations to the driving oscillation. The next term introduces damping $(-\psi)$ and a term which induces rotation of the phase of ψ at the subharmonic frequency ω(half the driving frequency). The complex coefficient for the Laplacian term is chosen to fit the known dispersion relation for the granular system being simulated. The next term ($|\psi|^2\psi$) is a nonlinearity which stabilizes the amplitude of the oscillations (next leading order term which respects $\psi \to -\psi$ symmetry). The last term ($-\rho\psi$) is the lowest order term that models an interaction between density and the order parameter.

As for the mass conservation equation, it is of the form $\partial_t \rho = \nabla \cdot (flux)$ with $(flux) = \alpha \rho \nabla |\psi|^2 + \beta \nabla \rho$. So, gradients in the density cost energy, as do gradients in the amplitude of the subharmonic oscillation. Grains then redistribute themselves to minimize the energy. The inclusion of mass conservation allows for the formation of square patterns in addition to stripe patterns (which can be observed with constant ρ).

Linear stability analysis shows that with these equations, the transition to squares (and to stripes) can be hysteretic for certain values of the parameters [5]. Oscillons were observed near the transition from the homogeneous phase to squares. The connection between the square pattern and oscillons is a consistent feature of experiments and models - it is a consequence of the symmetry under 180° changes in the phase ($\psi \rightarrow -\psi$). A square lattice can form composed of alternating positive and negative oscillons, which corresponds to the square phase.

One question is why these equations should not apply equally well to vibrated layer of water. Oscillons have not been observed in that system (though they have been observed in suspended clay). In the case of granular materials, it is possible for the spacing between grains to change by a large amount depending on the conditions at that point. Beyond a certain point, granular materials are incompressible (hard to make grains overlap) but they are certainly not in-expandable. Water however resists changes in volume very sharply, so the condition $\rho = const$ may be met quite well in that case. In the case of a suspension of clay, the local density may still change as the suspension could become more or less concentrated depending on the local conditions. It is also conceivable that it is simply a matter of the properties of water leading to the coefficients in the equations being slightly above (or below) the necessary threshold to form oscillons.

In [3], the oscillating grains are modelled as Faraday waves. Barashenkov, et. al. start with a lattice of coupled nonlinear oscillators which are experiencing a driving force at frequency 2ω. Then the behavior is expanded in terms of subharmonic (frequency ω) oscillations around the driving signal. This results in an amplitude equation:

$$i\partial_t\psi + \nabla^2\psi + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi$$

This equation is essentially the same as the amplitude equation from [5] (note that here γ is the damping and h the driving force, whereas in [5] γ was the driving force; also, the ω dependance has been factored out), with two differences. For one, there is no coupling to the local density (nor consideration of variations in local density), and secondly the laplacian term is not multiplied by a tunable complex value. This may end up to be very important in the dynamics, as it means that there is not a direct connection between the wavelength of a structure and its growth or damping-out (for $\psi = e^{ikx}e^{\omega t}$, it leads to $\omega = -ik^2 + ...$ rather than $\omega = -k^2 + ...$).

They find that there are two localized solutions to this equation which have a definite amplitude and phase (relative to the driving cycle), and a nodeless envelope function: $\psi^{\pm} = A_{\pm}e^{-i\theta_{\pm}}R(A_{\pm}r)$,$\nabla^2 R - R + 2R^3 = 0$[3]. The solutions for R which contain nodes are found to be unstable. This is already somewhat unusual, as the oscillons observed in experiments do have an oscillating envelope function (though not all solutions for R must be nodeless).

It is found (by looking at perturbations around the localized solutions) that one of these localized solutions is unstable for all values of the parameters (ψ^{-}, where $A_{-} = 1 - \sqrt{h^2 - \gamma^2}$ and $\theta_{-} = \frac{\pi}{2} - \frac{1}{2}sin^{-1}(\frac{\gamma}{h})$). However, the other solution ($A_{+} = 1 + \sqrt{h^2 - \gamma^2}$, $\theta_{+} = \frac{1}{2}sin^{-1}(\frac{\gamma}{h})$) is found to be stable for large values of the damping γ (and $h \geq \gamma$). This is consistent with the observation that oscillons appear for large (effective) viscosity in granular systems [4]. One thing that might be interesting to look into is whether the peak/crater behavior of the oscillons in granular systems leads the driving force by the angle predicted by this analysis.

The amplitude equation as written does not support the formation of square patterns [5], so this system is an interesting exception to the observed relation between square patterns and oscillons. It may be that the reason oscillons are not seen in water is related to the damping, and that a more viscous (but still incompressible) fluid might support them. It may be that this is related to why the oscillons predicted here have no nodes in their envelope function, whereas those experimentally observed (and those in other models) have an oscillating envelope. Barashenkov, et. al. analyze the decay of oscillons with nodal solutions and predict that an oscillon with one node should decay into five nodeless oscillons in a '+' pattern (by analyzing the instability of angular perturbations, and via numerical simulation). This would be something worth trying to reproduce experimentally, though I suspect that mass conservation (not explicitly present in this model) may require oscillons with at least one node, as the grains that compose the oscillon during a peak must come from the regions around that peak, and the grains that leave when the oscillon craters must then occupy surrounding regions.

2.2 Swift-Hohenberg Equation

The Swift-Hohenberg equation describes a system that forms 2D patterns of stripes and hexagons by introducing a preferred lengthscale [10]:

$$\partial_t \psi = R\psi - (\nabla^2 + 1)^2\psi + \alpha\psi^2 - \psi^3$$

The formation of patterns from the homogeneous solution occurs when the parameter R passes a threshold value (for fixed α). In this case, the transition (stripes and hexagons) is supercritical: there is no hysteresis. We can examine this by writing an amplitude equation for the striped pattern:

$$\psi = Ae^{ix} + c.c.$$

$$\partial_t A = (R + \alpha A - A^2)A \equiv 0$$

$$A(A^2 - \alpha A - R) = 0 \rightarrow A = \frac{\alpha \pm \sqrt{\alpha^2 + 4R}}{2}, A = 0$$

So for values of $R < -\frac{\alpha^2}{4}$, the only solution is $A = 0$. For $R > -\frac{\alpha^2}{4}$ there are two solutions which diverge from $A = \frac{\alpha}{2}$ at $R = -\frac{\alpha^2}{4}$. The case for hexagons is similar in derivation.

Oscillons have a symmetry under reversal of sign (peaks to craters). The hexagons pattern cannot respect that symmetry, so we cannot look for oscillons near a hexagonal phase. The α term in the Swift-Hohenberg equation breaks that reversal symmetry. In [2], terms were added in order to create a stable square pattern, and the α term was removed in order to preserve inversion symmetry. A higher order (ψ^5) term was also added in order to make the transition to squares subcritical (Fig. 2). That is, there is a region in phase space in which the square pattern can occur, but which a direct transition to the square pattern would involve a large-scale rearrangement of the system.

The resulting equation is [2]:

$$\partial_t \psi = R\psi - (\nabla^2 + 1)^2\psi + b\psi^3 - c\psi^5 + e\nabla \cdot [(\nabla\psi)^3] - \beta_1\psi(\nabla\psi)^2 - \beta_2\psi^2\nabla^2\psi$$

The extra gradient terms are added to stabilize square patterns. An amplitude equation for the square pattern can be derived by writing:

$$\psi = A(e^{ix} + e^{iy}) + c.c.$$

Substituting this into the equation for ψ, an equation of the form: $\partial_t A = f(A, R, b, c, e, \beta_1, \beta_2) \cdot A$ can be found, at which point stable values of A can be determined by setting the time derivative to zero. The stable values of A correspond to the solutions of a polynomial of the form $aA^4 + bA^2 + c = 0$ where a, b, c depend on the various parameters of the model. So as a result, one can determine for what values of the parameters the transition to squares will be subcritical [2].

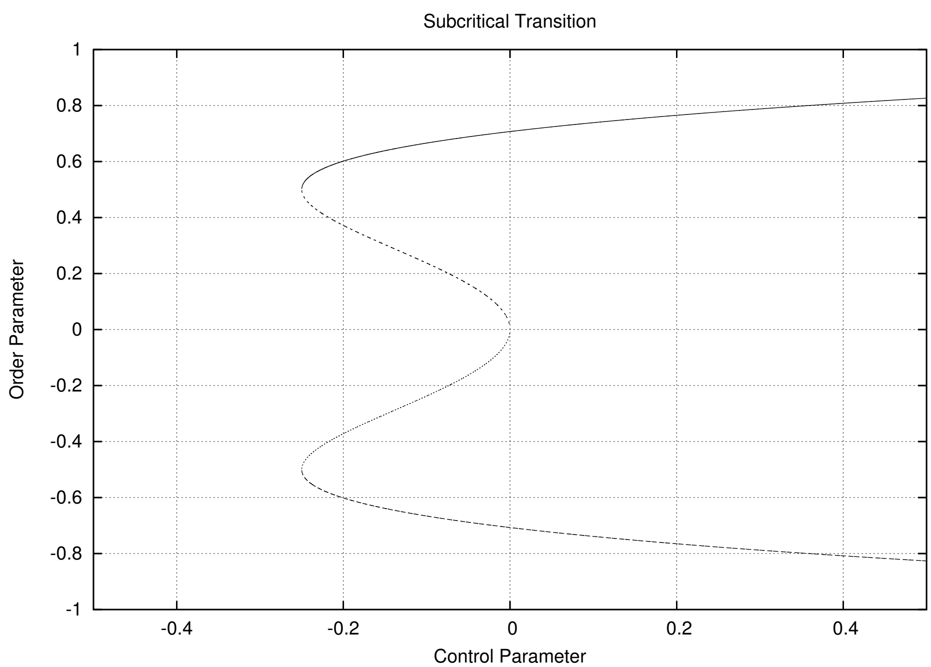

Figure 2: Subcritical transition. For values of the control parameter less than zero there is still a nonzero value of the order parameter. However, those values can only be reached (continuously) by ramping down from positive control parameter.

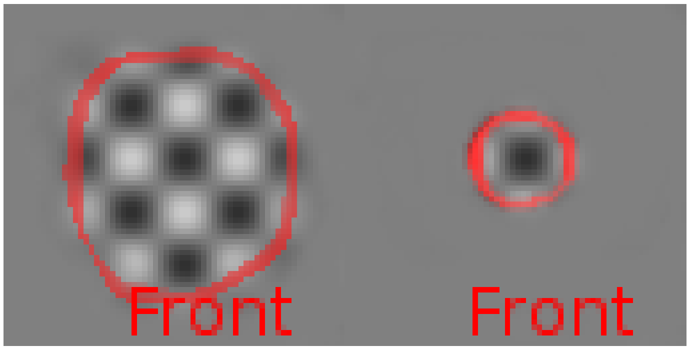

Figure 3: Idealized depiction of a localized square pattern.

Crawford and Riecke[2] proposed that oscillons are essentially single unit-cells of the square pattern which are surrounded by a stationary front (Fig. 3). This would require terms that stabilize the interface, as normally an interface will have nonzero velocity (which would prevent the occurance of a stable oscillon). Crawford and Riecke investigate pinning as a mechanism for the stabilization of interfaces (corresponding to the hysteresis added in the quintic term). They find that the scaling of the range of stability of a stationary front (range of R) matches a prediction of the expected functional form ($e^{-\alpha/\sqrt{R}}$) from [9]. In their simulations, Crawford and Riecke observe a case where a prepared square pattern decays into unevenly distributed oscillons as the control parameter is lowered past the transition (bright squares become bright oscillons, dark squares become dark oscillons). This seems to support their interpretation, especially as there is a similar observation in the original experiment [4] of an 'oscillon crystal' with the characteristics of the square pattern.

2.3 Other Physical Systems

The gas that forms the envelope of stars obeys a set of coupled hydrodynamic and radiative equations. The curvature of the star may usually be neglected (not a neutron star...) for a thin layer near the outer envelope. The resulting equations are [1]:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho(\partial_t + \mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho g \hat{\mathbf{z}}$$

$$C_v(\partial_t + \mathbf{u} \cdot \nabla)T + p\nabla \cdot \mathbf{u} = Q(T, \rho)$$

$$p = R\rho T$$

The equations are rotationally symmetric in the x and y directions, but not in the z direction. As a result, small oscillations away from equilibrium of these quantities can all be considered as slaved to a single time-and-space-dependant (complex) amplitude $A(x, y, t)$ and separately their **z** dependance (which may be different for the different variables).

The temperature is assumed to vary linearly with the **z** coordinate (the purely diffusive solution). The rate of variation is taken to be $\beta(\frac{z}{z_*})$ (z_* establishes a length-scale). As a result, the pressure and density are fixed by the equation of state and the condition $\mathbf{u} = 0$[1]:

$$\rho = \rho_0(\frac{z}{z_*})^m, P = P_0(\frac{z}{z_*})^{m+1}, m = \frac{g}{R\beta} - 1$$

The thickness of the layer is then determined by a number z_0, where the fluid considered lies in the region $z = z_0..1$. There is one other parameter to consider: the form of $Q(T, \rho)$ which describes the effect of absorption and emission on the local temperature. This function is Taylor-expanded about the equilibrium temperature, and the form is assumed to be restorative at first order (that is, the first order coefficient is negative: $Q_T = -\rho_0 C_v q$).

The resulting amplitude equation from expansion around equilibrium is of the form[1]:

$$\partial_t A = \mu A + \alpha \nabla^2 A + f(|A|^2)A$$

If $f(|A|^2)$ is extended to 2nd order (so a term $\partial_t A = ... - A^5...$) then oscillons exist in the system for some values of the coefficients in $f(|A|^2)$. The coefficients are related to the parameters z_0, m, q (and γ, the ratio of specific heats at constant pressure and volume). These can be related to the results for a particular stellar model, and as such it can be determined whether in that model oscillons would be a potential structure in acoustic fluctuations around hydrostatic equilibrium.

3 Conclusions

A common thread between many of the models which give rise to oscillons is that they occur on the border of a subcritical transition from a homogeneous phase to squares; they seem to be stabilized by the hysteresis in the transition. This has been seen in experiment as well, in the breakup of a square pattern into oscillons as the driving amplitude was decreased. However, there is at least one exception to this observation [3] which suggests that a more general explanation of the stabilization is needed.

There are several predictions for the shape, height, and phase leading of oscillons in [3]. It would be useful to test some of these predictions against experiments in granular materials and in water. Based on their model, it might be possible to observe oscillons in water if the damping could be decreased (perhaps by an additive such as glycerol). It would be useful to determine (from theory or experiment) why the experimentally observed oscillons have one node whereas the ones predicted by this model are nodeless (and non-nodeless oscillons are predicted to be unstable to decay into nodeless ones).

It is interesting that oscillons are observed in both inertial models and fully damped models (stellar oscillons, Swift-Hohenberg equation). However, in inertial models they appear only for sufficiently large values of the damping. So there is a question of whether or not the addition of a degree of freedom in the inertial models cause any qualitative change in oscillon behavior (beyond decreasing the stability of oscillons). How do bound groups of oscillons behave in inertial models when compared to damped models?

The model of stellar acoustic oscillons [1] is in terms of a set of parameters whose relation to the physical parameters of the star being studied is not trivial. It would be informative to see the results of calculating those model parameters for various types of stars so that one can determine if oscillons can only occur in stars of a certain radius, stellar type, age, and so on. That way, it would be more clear whether they can be expected to play some role in the dynamics of our own star (and as such be directly observable). The amplitude and width of such oscillons would also important in determining whether they can be observed experimentally or not (these can be determined by mapping onto the case in [3], in which it is possible to determine analytically the shape or by simulation). Also, how do the convective modes affect the stability of oscillons. As the determination of an amplitude equation was done with respect to hydrostatic conditions, the result may be quite different. Can oscillons only exist in the middle of convection cells (fairly constant flow), but not at the boundaries?

Another avenue of exploration in is simulating granular systems with hard spheres. It should be possible relatively soon to reach the compting power necessary to handle the $\approx 2 \times 10^5$ spheres necessary to observe oscillons. One aspect of this is that the evolution of the distribution of spheres can be compared to the continuum models of oscillons. Also, if the trajectories of spheres in an oscillon are mostly vertical, it suggests that redistribution of mass is not significant, whereas if the trajectories are circular, like a convection cell, it suggests that the opposite. Perhaps this could be tested experimentally using a few opaque tracer beads inside a bulk of transparent beads.

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