

Fractional charge in the fractional quantum hall system

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The Luttinger liquid is believed to be the emergent behaviors of 1D interacting electron gas on the edge of the fractional quantum hall (FQH) system. A brief summary of the Luttinger liquid is given. And tunneling of the Luttinger liquid between edges is shown to enable a direct measurement of the fractional charge of the quasiparticle.

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) has uncovered a number of unique and unexpected quantum behaviors of two-dimensional electrons. For the integer quantum Hall effect (IQHE), the conductance of the bulk is quantized as $\sigma_{xy} = n(e^2/h)$ for integer n . This behavior can be explained as the complete filling of a Landau level in the 2D free electron system with an external magnetic field. Since there is no energy gap when the system is in a half-filled Landau level, but a gap between 2 different Landau levels, it is easy to understand the integer quantization of the hall conductance in the quantum picture. On the other hand, the FQHE is comes from the strongly interacting electrons such that the conductance of the bulk is quantized as $\sigma = \nu(e^2/h)$ for a fractional number ν (e.g. $\nu=1/3, 1/5$). The fractional number of ν has a direct consequence that the system is stable when the Landau level is partially-filled (i.e. only ν of a level is filled). And this effect can not be explained in the weakly interacting electron picture and thus a new picture of strongly correlated electrons is required.

Similar to the BCS theory in superconductivity, a variational functional called Laughlin state is used as an functional state of the simple FQHE (simple means $\nu = 1/n$ for odd integer $n>1$, for example $\nu=1/3$). The Laughlin wavefunction ψ_0 is:

$$\psi_0 = \prod_{i<j} (z_i - z_j)^3 \exp\left(-\frac{1}{4} \sum_i |z_i|^2\right) \quad (1.1)$$

where $z_i = x_i + iy_i$ is the complex coordinate of the i -th electron. It is easy to show that the Laughlin wavefunction is a solution of an incompressible fluid equation.

$$|\psi_0(z_1, z_2, \dots, z_N)|^2 = e^{-\Phi} \quad (1.2)$$

it becomes:

$$\Phi = -\frac{2}{\nu} \sum_{i<j} \ln |z_i - z_j| + \frac{1}{2} \sum_k |z_k|^2 \quad (1.3)$$

which is the potential energy of a one component plasma with charge $q = \sqrt{2/\nu}$ moving in a neutralizing background of density $\rho = \nu/2\pi$. The first term is due to the Coulomb repulsion among the particles while the second term is due to the interaction of the particles with the neutralizing background. Thus the Laughlin wavefunction represents an incompressible quantum liquid and the incompressibility is the essence of the FQHE.

And so any excitation from the Laughlin state inside the bulk establishes an energy gap. To consider the low energy excitation of the Laughlin state, excitation on the edge (edge state) is the only possibility. And a theory of these edge states can be derived from a hydrodynamic formulation. In section II which is equivalent to the 1D Luttinger liquid.

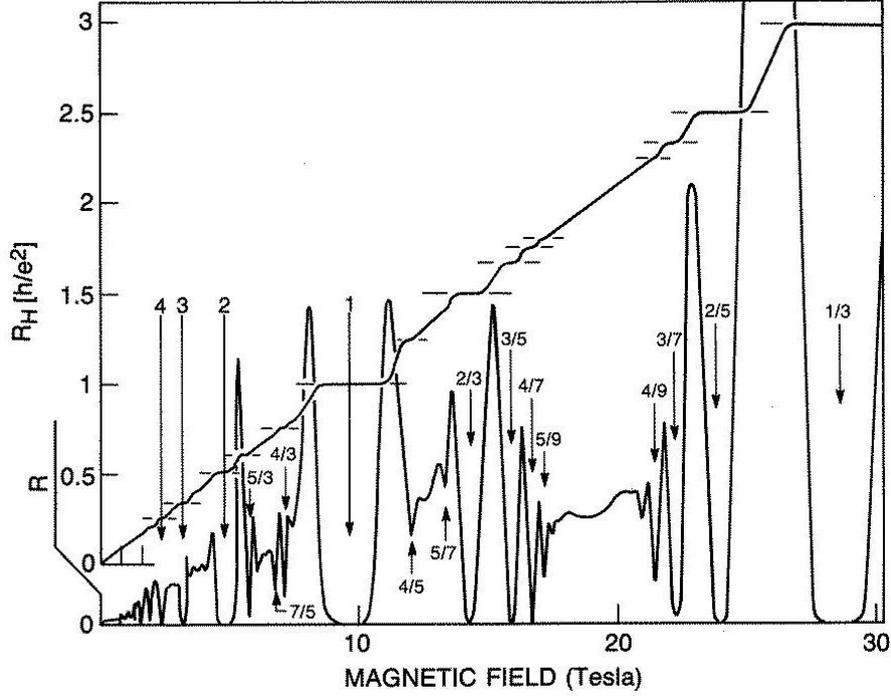


FIG. 1: The hall resistance measured in the 2D electron system with a perpendicular magnetic field. The graph shows that the hall coefficient is fractional quantized with the unit of h/e^2 . [7]

Similar to the BCS theory in superconductor, the Laughlin state is the variational function which can be used to describe the fractional quantum system. [1] Laughlin suggested that the quasiparticle in the FQHE carry fractional charge. Despite the overall success of Laughlin theory, a direct and convincing experimental measurement of fractional charge is necessary to verify this theory. So, the measurement of the fractional charge in the fractional quantum hall system becomes important.

This measurement is elusive. It is because the conventional dc transport measurements are insensitive to the charge carrier. In contrast, nonequilibrium noise measurement, such as shot noise, are dependent on the absolute magnitude of the carrier's charge. So, a nonequilibrium edge state transport, focusing on the fluctuations in the current and voltage, is proposed to measure the charge of the quasiparticle in FQH system and this experiment showed that the quasiparticle in FQHE has the fractional charge. [3][4]

II. LUTTINGER LIQUID

A. Hydrodynamic formulation

Like a classical incompressible fluid, it is expected the variation of the density on the edge is much easier than that inside the bulk. So, the existence of the edge excitation in the Laughlin is quite natural. Using the 1D edge density $\rho(x)$, the propagation of the disturbance is derived from the wave equation

$$\partial_t \rho(x) - v \partial_x \rho(x) = 0 \quad (2.1)$$

In an incompressible fluid, the vanishing of dissipation ($\sigma_{xx} = 0$) and the presence of a non-zero Hall conductance, generates a current along the edge due to the electric field which bounded the quantum fluid:

$$\vec{j} = \sigma_{xy} \hat{z} \times \vec{E} = -\sigma_{xy} \hat{z} \times \nabla V, \vec{E} = -\nabla V \quad (2.2)$$

With a vertical magnetic field ($\vec{B} = B\hat{z}$) and a y-direction confined electric field ($\vec{E} = -E\hat{y}$). The

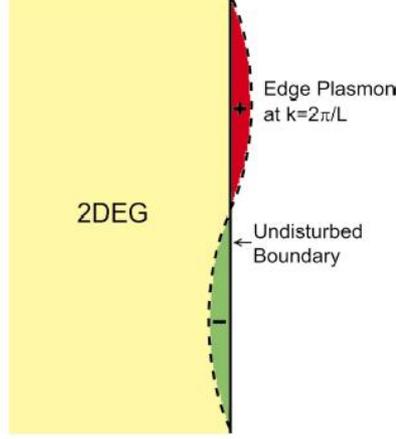


FIG. 2: Edge density plasmon wave at the boundary of a fractional quantum Hall fluid. In the absence of a density disturbance the boundary is depicted as a straight line. In the presence of a plasmon wave at wave vector $k=\Pi/L$, where L is the length of the boundary, some regions accumulate excess charge while others suffer a depletion of charge.

electron drift with the velocity $v = (E/B)c$. Introducing the vertical displacement $h(x)$ at position x along the edge, related to the edge density by

$$h(x) = \rho(x)n, n = \frac{eB}{hc} \quad (2.3)$$

when eB/hc is the density of a whole Landau level, and thus n is just the average density of the fractional Hall bulk. So, the Hamiltonian associated with the chiral edge wave become:

$$H = \int dx \frac{1}{2} e \rho(x) h(x) E = \pi \hbar \frac{v}{\nu} \int dx [\rho(x)]^2 \quad (2.4)$$

In Fourier space, it becomes:

$$H = \pi \hbar \frac{v}{\nu} \sum_{q>0} \rho_q \rho_{-q} \quad (2.5)$$

By equation 2.1, the corresponding wave equation in k space become:

$$\dot{\rho}_q = -i\nu q \rho_q \quad (2.6)$$

Consider the Hamilton's equation for the coordinates and momenta, we can identify the generalized coordinates and momenta as:

$$q_q = \rho_q, p_q = \frac{i\hbar}{\nu q} \rho_{-q} \quad (2.7)$$

Next step is the quantize the Hamiltonian by canonical quantization:

$$[q_q, p_{q'}] = i\hbar\delta_{q,q'} \quad (2.8)$$

giving

$$\begin{aligned} [\rho_q, \rho_{-q'}] &= \frac{\nu}{2\pi} q\delta_{q,q'}, \\ [H, \rho_q] &= -qv\rho_q \end{aligned} \quad (2.9)$$

The above algebra is called the (U(1)) Kac-Moody algebra. The low-lying charge excitation obviously correspond to adding (removing) electrons to (from) the edge. The charge excitation carry integer charge e and are created by electron operators ψ^+ where $\rho(x) = \psi(x)^+\psi(x)$ and $[\psi(x), \psi^+(x')] = \delta(x-x')$. The above theory of edge excitations is formulated in term of the 1D density operator $\rho(x)$. So the central question is to write the electron operator in terms of the density operator. The electron operator on the edge creates a localized charge should also satisfy:

$$[\rho(x), \psi^+(x')] = \delta(x-x')\psi(x') \quad (2.10)$$

since $\rho(x) = \psi(x)^+\psi(x)$. It can show that for ψ to satisfy all the above constraint. It is given by:

$$\begin{aligned} \psi(x) &\sim e^{i(1/\nu)\theta}, \text{ where } \partial_x\theta = 2\pi\hbar\rho \\ [\theta(x), \theta(x')] &= -i\pi\nu\text{sgn}(x-x') \end{aligned} \quad (2.11)$$

And the Hamiltonian becomes:

$$H = \frac{v}{2h\nu} \int dx (\partial_x\theta)^2 = \frac{1}{2\nu} \int dx (\partial_x\theta)^2 \quad (2.12)$$

by setting $v/h=1$ for simplicity. It turns out that

$$\psi(x)\psi(x') = (-1)^{1/\nu}\psi(x')\psi(x) \quad (2.13)$$

which imply $1/\nu$ must be an odd integer, which is the simple fractional quantum Hall. As a summary, it is known that Laughlin wavefunction is the variational ground state of the FQHE. And it satisfies the equation of incompressible hydrodynamics equation. Thus, it is argued that the low energy excitation is only possible on the edge and finally these low energy edge excitation is derived based on the model of classical incompressible hydrodynamics and quantize it finally.

This procedure which convert the fermion (electron $\phi(x)$), to the bosonic-like excitation ($\psi(x)$) is called bosonization. The quantum liquid with these bosonic excitations is called the Luttinger liquid, as opposed to the Fermi liquid.

There is a large difference between the Fermi liquid and the Luttinger liquid. In a more general formalism of the Luttinger liquid, it is a model of interacting fermions in 1D case and based on the linearization of the energy spectrum near the Fermi wave vector k_F . 1D is a special case because there are only 2 discrete fermi surfaces (points in 1D case). The low energy excitation is only possible for fermions with wave vector k with $|k - k_F| \ll 1$ or $|k - 2k_F| \ll 1$. This indicates that as the excitation energy approaches zero, there is a forbidden region for electron-hole excitations at $0 < k < k_F$, in contrast to the Fermi liquid in 2D or higher dimension.

As a result, the behavior between Fermi liquid and Luttinger liquid is very different. Some of the most significant consequences include the following:

(1) The one-to-one correspondence of the unperturbed k -electron state to the elementary excitations of the interacting system is lost in the Luttinger liquid.

(2) The discontinuity of the distribution function for the particle at the fermi surface is lost for the Luttinger liquid. It indicates that in contrast to the Fermi liquid, there is no sharp quasiparticle for $k = k_F$.

(3) Power law behavior in the correlation function in Luttinger liquid instead of the exponential one in Fermi liquid. It is not difficult to show that the Green's function of the electron operator becomes:

$$G(k, \omega) = \langle 0 | T \{ \psi^+, \psi \} | 0 \rangle \propto \frac{(vk + \omega)^{1/\nu-1}}{vk - \omega} \quad (2.14)$$

Since θ is a D=1+1 free massless field with a propagator

$$\begin{aligned} \langle 0 | \theta(x, t) \theta(0) | 0 \rangle &= \langle 0 | e^{iHt} \theta(x) e^{-iHt} \theta(0) | 0 \rangle \\ &= \langle 0 | e^{-v\partial_x t} \theta(x) \theta(0) | 0 \rangle \\ &= \langle 0 | \theta(x - vt) \theta(0) | 0 \rangle \\ &= -\nu \ln(x - vt) + \text{constant} \end{aligned} \quad (2.15)$$

and the electron Green's function can be calculated as

$$G(x, t) = \langle T \{ \psi^+(x, t) \psi(0) \} \rangle = \exp \left(\frac{1}{\nu^2} \langle \theta(x, t) \theta(0) \rangle \right) \propto \frac{1}{(x - vt)^{1/\nu}} \quad (2.16)$$

This Green's function leading to a power law of the electron tunneling density of states (DOS) with the exponent value of $1/\nu - 1$:

$$D(\omega) = \int_{-\infty}^{\infty} dk G(k, \omega) \propto \omega^{1/\nu-1} \quad (2.17)$$

For simple quantum fluid with $\nu = 1/3$, this theory predicts the value 2 for the exponent in the electron tunneling DOS. This is a central result and a dramatic prediction of the chiral Luttinger liquid model of edge dynamics in FQH.

B. Tunneling of current between edges

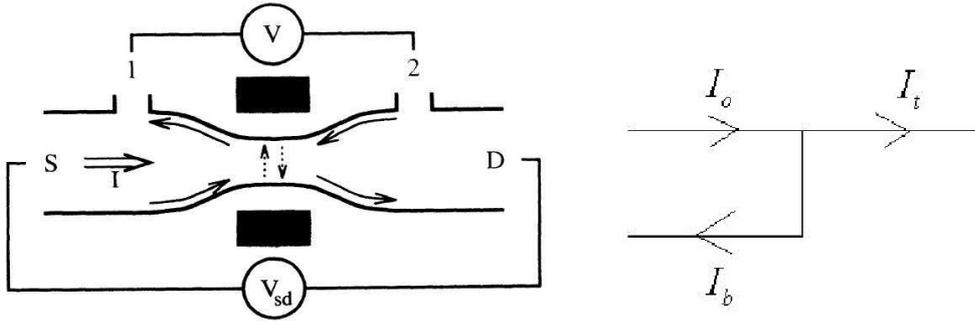


FIG. 3: (Left) Schematic of quantum Hall bar with gated constriction (G) separating source from drain. Of interest are nonequilibrium fluctuations in the current, I , and voltage, V , in the presence of an applied source-to-drain voltage, V_{sd} . [3] (Right) I_o is the incoming current, I_t is the transmitted current and I_b is the back scattering current in the quantum Hall bar.

In order to measure the charge of the quasiparticle in the FQH system, a non-equilibrium transport theory for the Luttinger liquid is proposed.[3] The set-up of the experiment is shown in fig. 3. A voltage difference, V_{sd} , is applied across a quantum Hall bar. In the presence of this source drain voltage, the top and the bottom edges which are incident from the drain and source are in thermal equilibrium at chemical potential separated by eV_{sd} . A gate G is applied at the middle of the Hall bar so that the edge states can tunnel from one to the other at where the gate located (For simple, we assume the gate is applied at $x=0$).

As the tunneling occurs, the backscattering of the edge state gives rise to a backscattered current, I_b and the transmitted current, I_t as shown in fig. 3. As a result, a nonequilibrium current and voltage fluctuation is established. Since the tunneling process involve one excitation each time, the charge of these excitations (νe) can be measured by the shot noise between lead 1 and 2. And it will be shown that the ratio of current noise to the voltage noise in the low frequency limit is simple proportional to the square of the charge of the quasiparticle, $(\nu e)^2$.

The fluctuation of current can be characterized by the correlation and response functions:

$$\begin{aligned} C_I(\omega) &= \frac{1}{2} \int dt e^{i\omega t} \langle \{I(t), I(0)\} \rangle \\ R_I(\omega) &= \frac{1}{2} \int dt e^{i\omega t} \langle [I(t), I(0)] \rangle \end{aligned} \quad (2.18)$$

and similar for the voltage. In equilibrium at temperature T , there are related by the fluctuation-dissipation theorem, $C(\omega) = \coth(\hbar\omega/2k_B T)R(\omega)$ but not the case in nonequilibrium.

To derive the relation between the correlation and the response functions, from section II A, we get the Lagrangian $L = 1/2\nu \int dx (\partial_x \theta)^2$ (2.12), which can be used to describe the nonequilibrium transport through the constriction (at $x=0$). The tunneling at $x=0$ is characterized by a perturbation term of weak scattering potential at $x=0$, $\lambda \psi^+(x=0)\psi(x=0)$. Integrating out of those fluctuations away from the constriction, i.e. for $x \neq 0$ and do the imaginary time transformation $t = i\tau$, an effective Lagrangian in Euclidean space in terms of $\theta = \theta(x=0, \tau)$ obtained:

$$S_E = \frac{1}{\nu} \int d\omega |\omega| |\theta(\omega)|^2 - \lambda \int d\tau \cos[2\sqrt{\pi}\theta(\tau)] \quad (2.19)$$

where λ is the amplitude for the tunneling of a Laughlin quasiparticle between edges and the current through the constriction is $I = \dot{\theta} / \sqrt{\pi}$, i.e. I_b .

To find the correlation and the response functions, the partition function is required which can be defined by:

$$S = \int D\theta_+ D\theta_- e^{S(\theta_{\pm})} \quad (2.20)$$

By the Keldysh approach, the real-time effective action can be expressed in terms of new field $\theta_{\pm} = \theta \pm (1/2)\tilde{\theta}$, and $S = S_0 + S_1 + S_2$ where:

$$\begin{aligned} S_0 &= -\frac{1}{\nu} \int d\omega \coth(\frac{\omega}{2T})\omega \left| \tilde{\theta}(\omega) \right|^2 + \frac{2i}{\nu} \int dt \tilde{\theta}(t) \dot{\theta}(t), \\ S_1 &= -i\lambda \int dt [\cos 2\sqrt{\pi}\theta_+ - \cos 2\sqrt{\pi}\theta_-], \\ S_2 &= \frac{i}{\sqrt{\pi}} \int dt \left[a(t) \dot{\tilde{\theta}}(t) + \eta(t) \dot{\theta}(t) \right] \end{aligned} \quad (2.21)$$

where η is a source field and the source-to-drain voltage is given by $V_{sd} = \dot{a}$. S_0 is the unperturbed term, S_1 is the tunneling term and S_2 is the coupling between the field with the external fields $a(t)$ and $\eta(t)$ such that the average current and the correlation function can be obtained easily by functional differentiation with respect to the external field, $\langle I \rangle = -i\delta \ln(Z)/\delta\eta$, and the correlation and response functions become:

$$C_I(\omega) = -\frac{1}{Z} \frac{\delta^2 Z}{\delta\eta(\omega)\delta\eta(-\omega)} \quad (2.22)$$

$$R_I(\omega) = -\frac{1}{Z} \left(\frac{\delta^2 Z}{\delta a(\omega)\delta\eta(-\omega)} - \frac{\delta^2 Z}{\delta\eta(\omega)\delta a(-\omega)} \right)$$

So, the only thing missed is the associated voltage operator \hat{V} which can be identified from the classical shot noise expression with charge νe :

$$\langle I \rangle = \frac{\nu e^2}{h} (V_{sd} - \langle V \rangle) \quad (2.23)$$

By differentiating $\ln Z$ with respect to the external field η , the voltage operator is identified as:

$$\hat{V} = (h/e)v \sin(2\sqrt{\pi}\theta + \nu a) \quad (2.24)$$

It is now straightforward to get the central result in the tunneling experiment. From equation 2.21 and 2.22, it turns out that:

$$C_I(\omega) - \coth\left(\frac{\hbar\omega}{2k_B T}\right) R_I(\omega) = \left(\frac{\nu e^2}{2\pi\hbar}\right) \left[C_V(\omega) - \coth\left(\frac{\hbar\omega}{2k_B T}\right) R_V(\omega) \right] \quad (2.25)$$

and in the low frequency limit, it becomes:

$$C_I(\omega \rightarrow 0) = \left(\frac{\nu e^2}{2\pi\hbar}\right) C_V(\omega \rightarrow 0) \quad (2.26)$$

In the limit of weak scattering where the derivations from perfectly quantized source-to-drain conductance are small, $\langle V \rangle \ll V_{sd}$, the dominant backscattering channel is the tunneling of the Laughlin quasiparticles and the information of the Laughlin quasiparticles can be obtained. In this limit, treat S_0 and S_2 as the unperturbed Lagrangian and S_1 as the perturbed term. The voltage noise and $\langle V \rangle$ can be obtained perturbatively in λ to second order and compare. The classical noise equation can be obtained:

$$C_V(\omega \rightarrow 0) = \frac{h}{e} \coth\left(\frac{\nu e V_{sd}}{2k_B T}\right) \langle V \rangle + \dots \quad (2.27)$$

Combine the above equations, it results:

$$C_I(\omega \rightarrow 0) = (\nu e)(I_{\max} - \langle I \rangle) = (\nu e)I_b \quad (2.28)$$

where $I_{\max} = \sigma_{xy} V_{sd} = (\nu e^2/h)V_{sd}$ is the maximum current tunnel between edges. So, by measuring the shot noise and the backscattered current, the charge of an individual quasiparticle can be obtained.

In summary, the Luttinger liquid based on the hydrodynamics formalism is presented, it shows that it is natural to get the Luttinger liquid Lagrangian in 1D edge state. Then, a nonequilibrium transport of the weak-tunneling quasiparticle is introduced by adding a local scattering potential in the Lagrangian. Integrating out the fluctuation away from the tunneling point, an effective theory is obtained. The correlation and the response functions can be obtained by the functional differentiation with respect to the external field. A simple relation between the shot noise and the backscattered current can be obtained finally and the fractional charge of the quasiparticle can be measured.

III. MEASUREMENT OF THE FRACTIONAL CHARGE

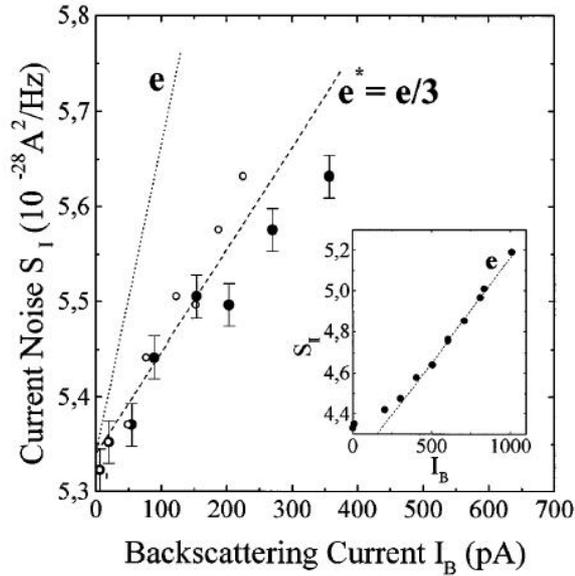


FIG. 4: Tunneling noise at $\nu = 1/3$ plotted versus I_b (filled circle) The slope for $e/3$ quasiparticles (dashed line) and electrons (dotted line) are shown. $T=25\text{mK}$. Inset: data in same units showing electron tunneling but in Integer Quantum Hall effect regime. The corresponding charge measured is e in this case is shown. $T=42\text{mK}$. [4]

In the last section, a method to measure the charge of the quasiparticles is described by the nonequilibrium shot noise measurement. And a simple relation between the current noise at low frequency limit and the backscattered current is shown:

$$C_I(\omega \rightarrow 0) = (\nu e)I_b \quad (3.1)$$

This experiment is done in 1997 [4] and the result is shown in fig. 4. FQHE with $\nu = 1/3$ is studied. From the graph and eqn 3.1, we can conclude that the quasiparticles have a charge of $e/3$ is carrying a current in the $\nu = 1/3$ FQH system. And it is an evident for the Laughlin theory and the validity of the 1D Luttinger liquid model used to describe the edge states in FQH.

IV. CONCLUSION

Theory of the Luttinger liquid is reviewed and it is able to model the edge state of the FQH bulk. A tunneling scheme between edge states on two different edges is studied. It showed that by measuring the shot noise of the system, the charge of the carrier can be found. Experiment on $\nu=1/3$ FQH system showed that the carrier's charge of the quasiparticle is $1/3e$, which is a strong evident on the Laughlin theory.

Another interesting question is about the fractional statistics of these quasiparticle. It is believed that these quasiparticle with the fractional charge also obey the fractional statistics (anyon) instead of boson or fermion statistics. That is, a phase factor between 0 and π comes out when we interchange two quasiparticles. (The phase factor is 0 for boson and π for fermion). For the simple FQH state, the phase factor is exactly equal to ν . But for the complicated FQH state, there is not a simple relation

between them. The measurement of the fractional statistics of these quasiparticles is interesting. And recently, some similar tunneling experiments of the edge states of FQH system is proposed to measure the fractional statistics directly.[8]

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