Scaling in Surface Growth

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May 2000

Physics 464 Spring 2000 Term Paper
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Abstract

The Kardar-Parisi-Zhang (KPZ) equation is a continuum model for surface growth sharing the same universality class with discrete version such as ballistic deposition and Eden growth model. We review scaling analysis and application of dynamic renormalization group to obtain exact scaling exponents from KPZ equation.

I. INTRODUCTION

Many physical phenomena consisting of an advancing interface, driven by chemical reaction or build up by a mass influx, appear to have similar morphology and growth dynamics. Examples include fluid flow in porous media, propagation of combustion fronts, and crystal growth by atomic deposition. See Fig. 1. Notwithstanding the varied details influencing each of these processes, scientists hope to extract a few common physical principles governing them. In the context of modern statistcal mechanics, we can study their scaling behavior and define *universality* classes. Analogous to the use of Ising model in critical phenomena, we shall introduce one that mimics nonequilibrium growth phenonmenon, use it to set the stage for scaling analysis and application of renormalization group (RG) techniques.

II. BALLISTIC DEPOSITION

The essential physics of some surface growth phenomena, such as colloidal aggregation and atomic deposition on thin films, is captured by the discrete *ballistic deposition* (BD) model [1,2,3]. In the BD model on a lattice, particles are released one at a time from randomly selected position above an initial surface. Particles follow straight vertical trajectories until they reach sites whose nearest neighbors are occupied. (Fig. 2). Some of the quantities needed to describe the growth process include:

$$\bar{h}(t) \equiv \frac{1}{L}\sum_{i=1}^{L} h(i,t) \tag{1}$$

where L is the system size – number of columns, and $h(i,t)$ is the height of column i at time t; and

$$w(L,t) \equiv \sqrt{\frac{1}{L}\sum_{i=1}^{L}(h(i,t)-\bar{h}(t))^2} \tag{2}$$

defined as the interface width which characterizes roughness of the interface using rms fluctuation in height. Simulation of BD model reveals scaling behavior of interface morphology and motivates hypothesis of the existence of scaling function. Fig. 3 is a sketch of the time dependence of interface width. It shows the interface width increases as a power of time, $w(L,t) \sim t^{\beta}$, with β as the growth exponent. The power law behavior would eventually enter a saturation regime in which the width attains a saturation value w_{sat}. The saturation value also follows a power behavior, $w_{sat} \sim L^{\alpha}$, α is called the roughness exponent. The time, τ, at which the system cross over to saturation regime depends on the system size: $\tau \sim L^{z}$, we call z the dynamic exponent. Near the crossover point $w(\tau) \sim \tau^{\beta}$ and $w(\tau) \sim L^{\alpha}$, as such, $\tau^{\beta} \sim L^{\alpha}$. The last relation implies that the exponents are not independent but satisfy the scaling law

$$z = \frac{\alpha}{\beta} \tag{3}$$

It is possible to collapse the data from different system sizes (L) by plotting $w(L,t)/w_{sat}(L)$ vs. t/τ [4], hence we can write the scaling relation

$$w(L,t) \sim L^{\alpha} f\left(\frac{t}{L^{z}}\right). \tag{4}$$

There is an important feature of the BD growth process worth mentioning: the correlation that develops along the surface. The correlation is due to the nearest neighbor sticking rule which tends to propagate overhang in the lateral direction. This is the mechanism that leads to saturation in a system of finite size as indicated by the relation $w_{sat} \sim L^{\alpha}$. For one dimensional interface, numerical simulations [1,3] show

$$\alpha = 0.47 \pm 0.02; \qquad \beta = 0.33 \pm 0.006. \tag{5}$$

We have to be aware of the fact that BD is modeling a nonequilibrium system, so unlike the Ising model we cannot possibly write down a partition function and hope to solve for the various exponents. But in the spirit of Landau theory, we can exploit symmetries of the problem and write down a coarse-grained continuum equation for surface growth. Solving the equation may help us gain further insight to the dynamic phenomenon.

III. KARDAR-PARISI-ZHANG EQUATION

Kardar, Parisi, and Zhang [5] proposed a continuum growth equation corresponding to the discrete BD model. We should expect the growth equation to have the form

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \Phi(h,\mathbf{x},t) + \eta(\mathbf{x},t). \tag{6}$$

$\Phi(h,\mathbf{x},t)$ is some general function and $\eta(\mathbf{x},t)$ is a noise term that reflects the random fluctuations in the deposition process. We can use symmetry arguments to construct terms needed for the $\Phi(h,\mathbf{x},t)$ function:

- *Time invariance.* Thus $\Phi(h,\mathbf{x},t)$ cannot have explicit time dependence.
- *Translation invariance along growth direction.* This implies $\Phi(h,\mathbf{x},t)$ must be constructed from combinations of ∇h, $\nabla^2 h$, ... , $\nabla^n h$.
- *Translation invariance along direction normal to growth direction.* Explicit $\mathbf{x}$ dependence is denied and only terms like $\partial^n/\partial \mathbf{x}^n$ are allowed.
- *Rotation and inversion symmetry about growth direction.* Odd order spatial derivatives such as ∇h, $\nabla(\nabla^2 h)$ are ruled out.

The simplest equation satisfying all the above symmetries is

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t) \tag{7}$$

Simple geometric argument will reveal that in order to incorporate the presence of lateral growth in ballistic deposition the nonlinear term $(\nabla h)^2$ must be included. In the hydrodynamic limit, all higher order derivatives make little contribution and are therefore ignored. Equation 7 has come to be known as the Kardar-Parisi-Zhang (KPZ) equation. Scale transformation $\mathbf{x} \to b\mathbf{x}$, $h \to b^\alpha h$, and $t \to b^z t$ turn Equation 7 into

$$b^{\alpha-z}\frac{\partial h}{\partial t} = \nu b^{\alpha-2}\nabla^2 h + \frac{\lambda}{2} b^{2\alpha-2}(\nabla h)^2 + b^{-d/2-z/2}\eta. \tag{8}$$

Note that the noise term is assumed to have zero average and no correlation. As such,

$$\langle \eta(b\mathbf{x}, b^z t)\eta(b\mathbf{x}', b^z t')\rangle = 2D\delta^d(b\mathbf{x} - b\mathbf{x}')\delta(b^z t - b^z t'). \tag{9}$$

Multiplying Equation 8 with $b^{z-\alpha}$ yields

$$\frac{\partial h}{\partial t} = \nu b^{z-2}\nabla^2 h + \frac{\lambda}{2} b^{\alpha+z-2}(\nabla h)^2 + b^{-d/2+z/2-\alpha}\eta. \tag{10}$$

If the interface is self-affine, the growth equation must be invariant under this transformation. That is, each term on the right hand side of Equation 10 must be independent of b; consequently,

$$z - 2 = 0 \tag{11}$$

$$\alpha + z = 2 \tag{12}$$

$$-\frac{d}{2} + \frac{z}{2} - \alpha = 0. \tag{13}$$

The above equations constitute an overdetermined system for α and z which is untenable. If we reason that the nonlinear term dominates over surface tension and decide to abandon $\nu\nabla^2 h$, we obtain

$$\alpha = \frac{2-d}{3} \tag{14}$$

$$\beta = \frac{2-d}{4+d}. \tag{15}$$

For $d = 1$, they yield $\alpha = 1/3$ and $\beta = 1/5$, which still do not tally with Equation 5. It turns out that the coefficients (ν, λ, D) are coupled and change under rescaling. Fluctuation-dissipation theorem can allow us to calculate the roughness exponent [5,6], but in this discussion we shall introduce the RG method to overcome the difficulty.

IV. RENORMALIZATION GROUP APPROACH

The key step in RG procedure is making adjustments to coupling constants $\mathbf{K} \equiv (K_1, K_2, ..., K_m)$ in the Hamiltonian after each block spin transformation. Each RG transformation yields

$$\mathbf{K}^{n+1} = \mathcal{R}(\mathbf{K}^n). \tag{16}$$

Repeated transformations produce a flow in the parameter space of coupling constants. If successfully applied, RG iteration will lead to an invariant Hamiltonian. In other words, there is a fixed point defined by

$$\mathbf{K}^* = \mathcal{R}(\mathbf{K}^*). \tag{17}$$

It is this fixed point that would enable us to determine scaling exponents. There is one qualification: the fixed point provides *equilibrium* properties of the system. We are pursuing *dynamic* properties of a growth process which is not in equilibrium. So modification have to be made before applying RG to the KPZ equation [7,8]. The basic idea is to solve the KPZ equation by perturbative expansion. Firstly, Fourier transform the KPZ equation to obtain

$$h(\mathbf{k}, \omega) = \frac{1}{\nu k^2 - i\omega}\eta(\mathbf{k}, \omega) + \lambda\mathcal{N}[h(\mathbf{k}, \omega)], \tag{18}$$

where $\mathcal{N}[h(\mathbf{k}, \omega)]$ is a functional integral of height h. Assuming that λ is a small parameter, we perform the iteration

$$\tilde{h}(\mathbf{k}, \omega) = \frac{1}{\nu k^2 - i\omega}\eta(\mathbf{k}, \omega) + \lambda\mathcal{N}\left[\frac{1}{\nu k^2 - i\omega}\eta(\mathbf{k}, \omega) + \lambda\mathcal{N}\right]. \tag{19}$$

The equation contains integrals over the phase space that diverge for small $\mathbf{k}$, and the RG procedure must include steps to integrate out fast modes from the growth equation and rescale the lattice. The flow equations describing the change in the parameters under RG transformation are

$$\frac{d\nu}{dl} = \nu\mathcal{R}_\nu(\nu, \lambda, D) \tag{20}$$

$$\frac{dD}{dl} = D\mathcal{R}_D(\nu, \lambda, D) \tag{21}$$

$$\frac{d\lambda}{dl} = \lambda\mathcal{R}_\lambda(\nu, \lambda, D). \tag{22}$$

The parameter l is defined by $dl = d\log b$. The full expression obtained from RG is

$$\frac{d\nu}{dl} = \nu\left[z - 2 + K_d g^2 \frac{2-d}{4d}\right] \tag{23}$$

$$\frac{dD}{dl} = D\left[z - d + 2\alpha + K_d\frac{g^2}{4}\right] \tag{24}$$

$$\frac{d\lambda}{dl} = \lambda \left[\alpha + z - 2 \right]. \tag{25}$$

The coupling constant g is defined as

$$g^2 \equiv \frac{\lambda^2 D}{\nu^3}, \tag{26}$$

where $K_d \equiv S_d/(2\pi)^d$, S_d here is the surface area of the d-dimensional unit sphere. The exponents can be obtained by seeking for the points at which

$$\frac{d\nu}{dl} = \frac{dD}{dl} = \frac{d\lambda}{dl} = 0. \tag{27}$$

The flow of the coupling constant is then

$$\frac{dg}{dl} = \frac{2-d}{2} g + K_d \frac{2d-3}{4d} g^3. \tag{28}$$

When $d = 1$, setting the above equation to zero yields two fixed points:

$$g_1^* = 0 \qquad g_2^* = \left(\frac{2}{K_d} \right)^{1/2}. \tag{29}$$

From the nonzero attractive fixed point we find that

$$z = \frac{3}{2} \qquad \alpha = \frac{1}{2} \tag{30}$$

the exact values of scaling exponents. When $d > 1$, we have to concede that perturbative methods fail in the strong coupling regime, i.e. $g > g_2^*$, and other numerical techniques have to be employed.

V. DISCUSSION

Despite of the success of KPZ theory in determining exact exponents for $d = 1$, experiments such as propagation of combustion front and fluid flow in paper have failed to support the prediction [9,10,11]. In these experiments, the velocity of propagation is usually affected by spatial inhomogeneities in the medium – quenched noise. Such irregularities are believed to generate anomalies in the exponent. This suggests the nature of the noise term in growth equations may play an important role in determining universality classes.

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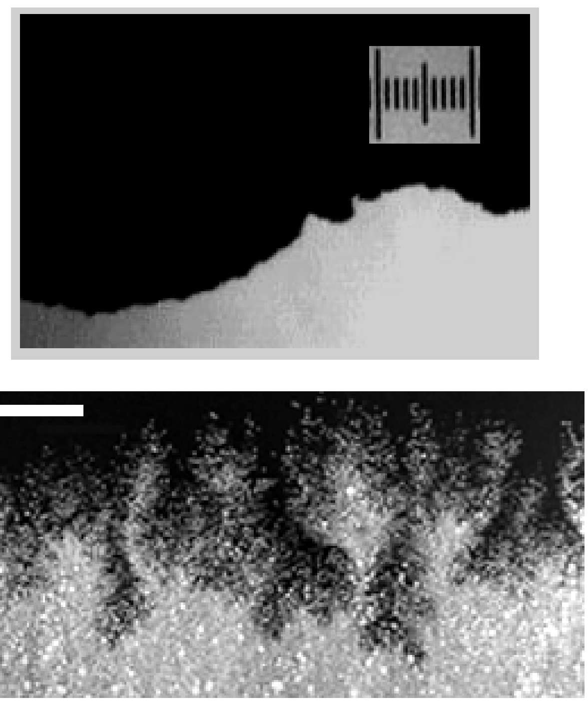

Figure 1: The top picture shows rough interface of the burning front of a piece of copier paper; the marker is $10\mu m$. The bottom picture is a simulation of 2D fluid flow in porous medium by trapping soap foam in a Hele-Shaw apparatus; the marker is 10cm.

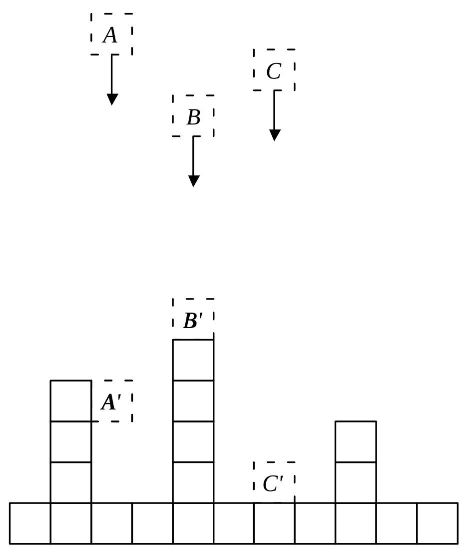

Figure 2: Ballistic Deposition: Particles stick to the first site along its trajectory that has an occupied nearest neighbor; after deposition A', B' and C' become part of the interface.

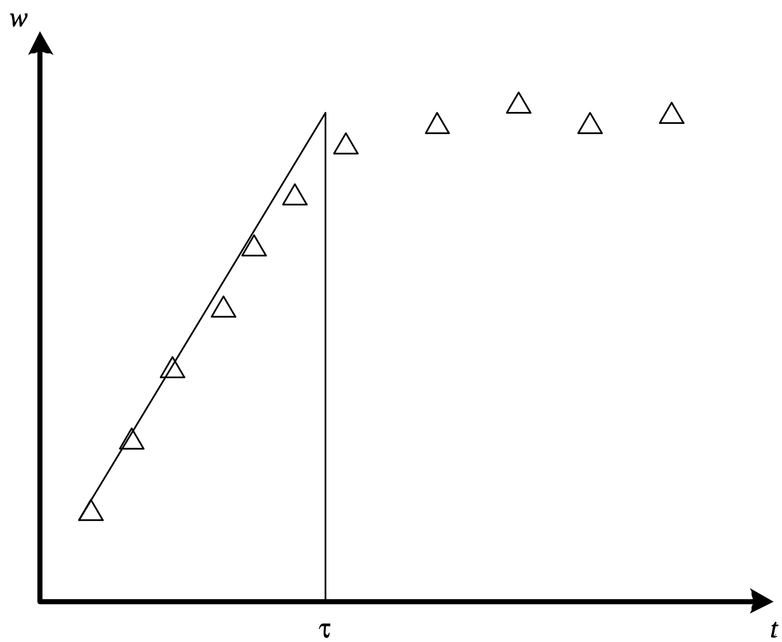

Figure 3: A sketch of the scaling behavior of BD model.