Log-periodic Scaling in Dynamical Systems

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Abstract

Heterogeneous systems undergoing rupture are similar to critical phenomena in that before they occur, long-range correlations are observed at many scales that cascade to events at larger scales. The precursory phenomena exhibit power law scaling near critical point, decorated by log-periodic corrections which are signatures of the underlying hierarchical structure of the system. The origin of these corrections is discussed, as well as its application as a predictive tool of time-of-failure of these systems, with special emphasis on earthquakes and financial crashes.

1 Introduction and Motivation

The prediction of catastrophes has been one of the most challenging problems faced by the physical sciences. This problem is important because of the great human and economic costs involved. However, the randomness inherent in the occurrence of these events makes their prediction seemingly impossible.

Recently however, researchers have come up with a general methodology for the scientific prediction of such events, based on the concepts and techniques of statistical and nonlinear physics. The key idea in this approach is that rupture phenomena are like critical phenomena, which are understood quite well in the physical sciences. One known result is that precursory phenomena follow power laws near rupture point. However, it has been proposed by these researchers that there is a log-periodic correction to this power law for heterogeneous systems like earthquakes and complex materials near critical point, and that by observing these log-periodic oscillations, it is possible to predict the time-of-failure of the system. This has been applied successfully to diverse systems like earthquakes, stock markets, fracture in materials, and population growth.

In this paper, we review the concept of log-periodic scaling and some of its applications to prediction. We begin by discussing the origins of log-periodic scaling in a simple and elegant concept called Discrete Scale Invariance, followed by the relation of this to critical phenomena. Then the applications of this method are shown to some representative systems, along with a discussion of the physics involved in the problems. In this paper, stock markets and earthquakes are discussed in detail. The paper is concluded with a brief overview of future challenges.

2 Discrete Scale Invariance

It shall be seen in this section that log-periodic corrections are just signatures of a hierarchical structure. The following discussion has been taken from [1], which is a review of current and future work in DSI, and the reader is encouraged to look up the article and the references therein.

Continuous scale invariance means reproducing itself on different time/space scales. That is, if we have an observable $\mathcal{O}$ which depends on a parameter x, then $\mathcal{O}$ is scale invariant under the arbitrary change $x \to \lambda x$ if there is a number μ such that

$$\mathcal{O}(x) = \mu \mathcal{O}(\lambda x) \tag{1}$$

The solution to the above is just a power law $\mathcal{O}(x) = Cx^{\alpha}$ with $\alpha = -\frac{\log \mu}{\log \lambda}$. Power laws are the hallmark of scale invariance (for a function of a single variable), because then the ratio of the observable at different scales does not depend on x:

$$\frac{\mathcal{O}(\lambda x)}{\mathcal{O}(x)} = \lambda^{\alpha} \tag{2}$$

Discrete Scale Invariance (DSI) is a weaker scale invariance where the observable obeys scale invariance only for specific choices of λ. The infinite but countable set of values λ_1, λ_2, etc. can be written as $\lambda_n = \lambda^n$.

To see how DSI leads to log-periodic corrections to scaling, we take the example of a triadic Cantor set, which is built recursively as follows: the first step is to divide the unit interval into 3 intervals, and delete the central one. Then each of the two remaining intervals is divided into 3 intervals (of length $\frac{1}{9}$), from which we delete the central interval, and so on. We see that the number of intervals grows as 2^n and the interval length shrinks as 3^{-n} at the nth iteration.

The triadic Cantor set is geometrically identical to itself only under coarse-graining by factors of arbitrary powers of 3. That is, if we take another magnification value, say 2, we cannot superimpose the magnified part on the initial Cantor set. Thus this set has the property of DSI under the fundamental scaling ratio 3.

Let $N_x(n)$ be the number of intervals at the nth iteration of the construction. Let x be the magnification factor. When the magnification increases by a factor 3, $N_x(n)$ increases by a factor 2 independent of the number n. The fractal dimension is defined as

$$D = \lim_{x \to \infty} \frac{\ln N_x(n)}{\ln x} = \frac{\ln 2}{\ln 3} = 0.63 \tag{3}$$

If we increase the magnification continuously from $x = 3^p$ to $x = 3^{p+1}$, the number of intervals in all classes jump by a factor of 2 at $x = 3^p$, but remain unchanged till the next jump by the same factor at $x = 3^{p+1}$. In the interval $3^p < x < 3^{p+1}$, the number $N_x(n)$ does not change, so the measured fractal dimension $D(x) = \frac{\ln N_x(n)}{\ln x}$ decreases. The fractal dimension D is obtained only when x is a power of 3. Thus, we have

$$N_x(n) = N_1(n) x^D P\Big(\frac{\ln x}{\ln 3}\Big) \tag{4}$$

where P is a function of period unity. We can expand P in a Fourier series

$$P\Big(\frac{\ln x}{\ln 3}\Big) = \sum_{n=-\infty}^{\infty} c_n exp\Big(2n\pi i \frac{\ln x}{\ln 3}\Big) \tag{5}$$

Plugging this expression back into (4), it appears that the exponent D is replaced by

$$D_n = D + ni\frac{2\pi}{\ln 3} \tag{6}$$

If we keep only the first term in the Fourier expansion of P and insert back in (4) we get

$$N_x(n) = N_1(n) x^D \Big(1 + 2\frac{c_1}{c_0} \cos\Big(2n\pi \frac{\ln x}{\ln 3}\Big)\Big) \tag{7}$$

where we have used $c_{-1} = c_1$, to ensure that $N_x(n)$ is real. Thus we observe a log-periodic correction to the leading power law behaviour due to the imaginary part of the exponent.

2.1 Critical phenomena

To make the connection of the above discussion with critical phenomena, we note that Eqn. (1) corresponds to a linearization, close to a fixed point, of a *discrete* RG equation describing the behaviour of the observable under a rescaling by an arbitrary factor λ [2].

$$K' = \phi(K) \tag{8}$$

$$F(K) = g(K) + \frac{1}{\mu} F(\phi(K)) \tag{9}$$

Close to K_c, we can linearize:

$$K' - K_c \approx \lambda(K - K_c) \tag{10}$$

where $\lambda = \frac{d\phi}{dK}|_{K_c} > 1$. We get

$$f(K) = \sum_{n=0}^{\infty} \frac{1}{\mu^n} g[\phi^{(n)}(K)] \tag{11}$$

where ϕ^n is the nth iterate of the transformation. We can clearly see that the sum above is singular at $K = K_c$. If we consider the kth derivative of f in Eqn. (11), we get a term which goes as $(\lambda^k/\mu)^n$ which is greater than 1 for large enough k, so the sum diverges. So, close to K_c, one has

$$f(K) \propto (K - K_c)^m \tag{12}$$

Looking at the singular part of the free energy in Eqn. (11), we get $\lambda^m = \mu$, and we also note that we have no conditions on m being a real number, so we get

$$m_n = \frac{\ln \mu}{\ln \lambda} + ni\frac{2\pi}{\ln \lambda} \tag{13}$$

which gives us a complex critical exponent as before.

We can therefore write a modified equation of the form

$$\epsilon(t) = A + B(t_f - t)^{\alpha}\Big[1 + C\cos\Big(2\pi\frac{\log(t_f - t)}{\log \lambda} + \psi\Big)\Big] \tag{14}$$

where t_f is the time-of-failure, which one wishes to predict, and ϵ is a measure of 'seismic' release.

3 Applications

It is realized that DSI and its associated log-periodicity may appear in natural systems spontaneously, i.e., without a pre-existing hierarchy as in the Cantor set. Examples are rupture in heterogeneous systems, earthquakes, and stock markets among many other systems. This may also be relevant to turbulence, where it is expected that there may be a preferred ratio in the cascade from large eddies to small ones. However, log-periodic oscillations have not been convincingly demonstrated in this field yet. (See [1] for more details)

We see that the period of the log-periodicity (in log-scale) is directly related to the existence of a preferred scaling ratio. Thus, we should interpret log-periodicity as the existence of a set of preferred length scales forming a geometric series in λ. This is in contradiction to the notion that a critical system exhibiting scale invariance has an infinite correlation length, hence only the microscopic cutoff(the lattice spacing) and the macroscopic cutoff (the size of the system) appear as distinguishable length scales. Thus this can be used to extract the relevant length scale for a system undergoing rupture.

In addition to this and the interesting notion of complex critical exponents, log-periodic oscillations can be used to predict the time-of-failure. In essence, all we have to do for this is to fit Eqn. (14) to the data we have. In this section, however, we shall dwell a bit on the nature of the systems we discuss, namely, the earth's crust and stock markets, to try and see where DSI is introduced.

3.1 Large Financial Crashes

It has been hypothesized by some researchers [3, 4] that stock market crashes are analogous to critical points . In doing so, three major points have been found

- It is possible to build a microscopic model of the stock market exhibiting well-defined critical points constrained to the limits of rational expectation, which is intuitively appealing, though it is argued that the number of irrational behaviour patterns outnumbers the rational ones.
- The predictions are relatively robust to model misspecification.
- These predictions are strongly borne out in the large stock market crashes of 1929 and 1987 (as also the recent Nasdaq crash in 2000).

3.1.1 Model

We review the model proposed in [3], in which the price dynamics is given by that of an asset that pays no dividends. We also ignore interest rates, risk aversions, and information asymmetry. In this very stylized framework, we have martingale pricing which states that the present price of the asset is equal to the future expectation conditional on information revealed up to the present. We can introduce the chance of a crash described by a hazard rate $h(t)$, which is the probability per unit time that the crash will happen in the next instant if it has not happened yet. We see that the higher the probability of a crash, the faster the price must increase in order to induce the investor to hold the asset. It is to be noted that the probability of the crash is an exogenous variable, and is not affected by the prices, something which may not sound satisfactory.

A crash happens when a large group of agents place sell orders simultaneously to create a large enough imbalance in the market. But these agents do not know each other, so it is proposed that all traders are organized into a network through which they influence each other locally. Thus only two 'forces' influence an investor: the opinions of his/her k nearest neighbours, and his/her own idiosyncracies. It is assumed that the investor imitates the neighbours. Thus the first term creates order, and the second disorder. And it is order which leads to the crash, in opposition to the popular notion that a crash is chaotic. We can make a mean-field approximation where the hazard rate evolves due to a collective result of interactions.

If we assume that each agent can only be in a state of buying or selling, then the above model is just like the zero-temperature random field Ising model with time dependent coupling constant:

$$s_i = sign(K \sum_{j \in N(i)} s_j + \sigma\epsilon_i + G) \tag{15}$$

where s_i is the state (1,−1) of the agent, K is the (positive) neighbour interaction, the sum runs over nearest neighbours, ϵ_i is the idiosyncracy(disorder) term, and G is the global influence (the external magnetic field). We can define analogs of magnetization and susceptibility too. We know that the susceptibility is related to the correlations, so we can interpret susceptibility as a measure of the ability of a system of agents to agree on an opinion. It is this emergence of global synchronization from local imitation that causes a crash. Thus we posit that the hazard rate of the crash follows a similar behaviour as susceptibility.

We do not know the equation that determines K as a function of time. However, all we assume is that K varies slowly. Then, prior to the critical date, we can use $K_c - K(t) \approx$ constant $\times(t_c - t)$. It is to be stressed that t_c is not the time of the crash, which we obviously cannot know by our rational expectation model, but the most probable time of crash.

3.1.2 DSI in Finance

To see how the concept of Discrete Scale Invariance enters the model in [3] discussed above, we place the network of agents on a lattice. We consider two such cases

2-Dimensional Lattice This is just equivalent to a 2D Ising Model. We thus have

$$\chi \approx A(K_c - K)^{-\gamma} \tag{16}$$

and

$$h(t) \approx B(t_c - t)^{-\alpha} \tag{17}$$

where the exponent α lies between 0 and 1 for economic reasons. From the martingale hypothesis, we get for the price before a crash:

$$\log \frac{p(t)}{p(t_0)} = \kappa \int_{t_0}^{t} h(t')dt' \tag{18}$$

where κ denotes the fraction of asset price lost in a crash. Solving, we get

$$\log p(t) \approx \log p_c - \frac{\kappa B}{\beta}(t_c - t)^{\beta} \tag{19}$$

where $\beta = 1 - \alpha$, and p_c is the price at the critical time. Thus we see, that α has to lie between 0 and 1. It is important to see that depending on whether the size of the crash is proportional to the price itself or that of the increase due to the bubble, either the logarithm of the price or the price itself should be the quantity characterizing the bubble.

Hierarchical Diamond Lattice The plane network we used in the last paragraph is a gross oversimplification. In the real world, it is highly unlikely that each agent will have the same number of neighbours. We have an ensemble of interacting units, which range in size over many orders of magnitude, from individual investors to massive professional investors like pension funds, to global economies. This does not imply that the stock market has any strict hierarchical structure, but there are a number of qualitatively hierarchical structures in society, so we can model our system in that way.

To make this network, we take a pair of traders linked to each other, and replace this link by a diamond where the two traders occupy two diametrically opposed vertices, and where the other two vertices are occupied by two new traders. Now we replace each of the existing four links by a diamond, in the same way and iterate the operation. After p iterations, most traders have only two neighbours, a few have 2^p ones, and the others are intermediate. This may be a more realistic model of the network.

Thus, we see that our model, through its hierarchy, implies DSI, and hence we have the associated signatures, namely, complex critical exponents and log-periodic corrections.

$$\log[p(t)] \approx \log[p_c] - \frac{\kappa}{\beta}(B_0(t_c - t)^\beta + B_1(t_c - t)^\beta \cos[\omega \log(t_c - t) + \phi]) \qquad (20)$$

3.1.3 The Nasdaq Crash of April 2000

In this section, we present the findings in [5] relating to a recent financial crash. The Nasdaq Composite is an index consisting mainly of the tech-stocks or the "New Economy", which lost more than 35% of its all-time high reached in early March this year. In figure 1, we see the logarithm of the Nasdaq Composite fitted with equation (14). The first point of the fit is the lowest value of the index prior to the crash, and the last point is the high. The details of the methodology are in [6, 7] . The best and third best fits are shown. The best fit predicted a crash date of 2 May and the third best fit predicted 31 March. The index lost most of its value in the week ending 14 April. It has been noted in other large crashes (1987, 1994, 1997), that the market crash started at a point between the last point and the predicted t_c, showing that the prediction may indeed be possible with Eqn. (14).

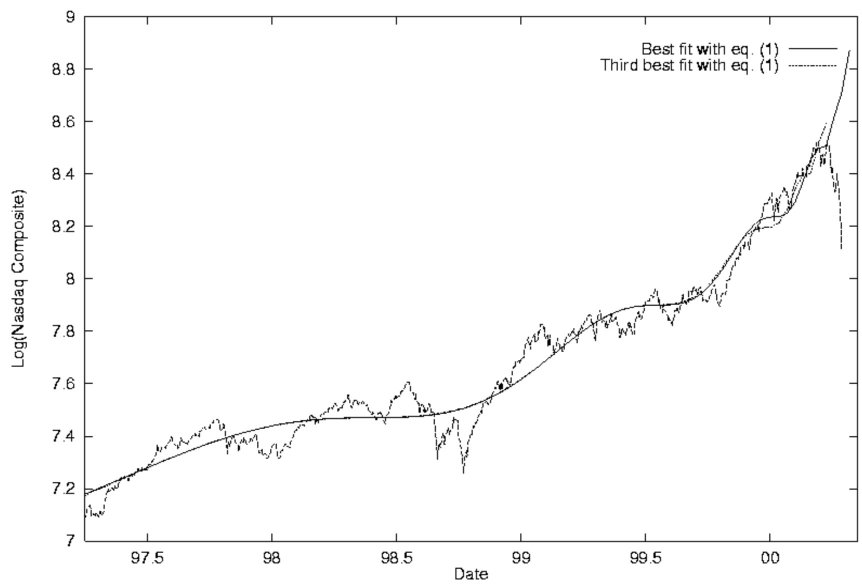

Figure 1: Best and third-best r.m.s. fits with Eqn. (14) to the natural logarithm of the Nasdaq Composite.

Furthermore, the crashes of large companies like IBM, and P & G may also be taken as precursors of a pending crash. (Figure 2)

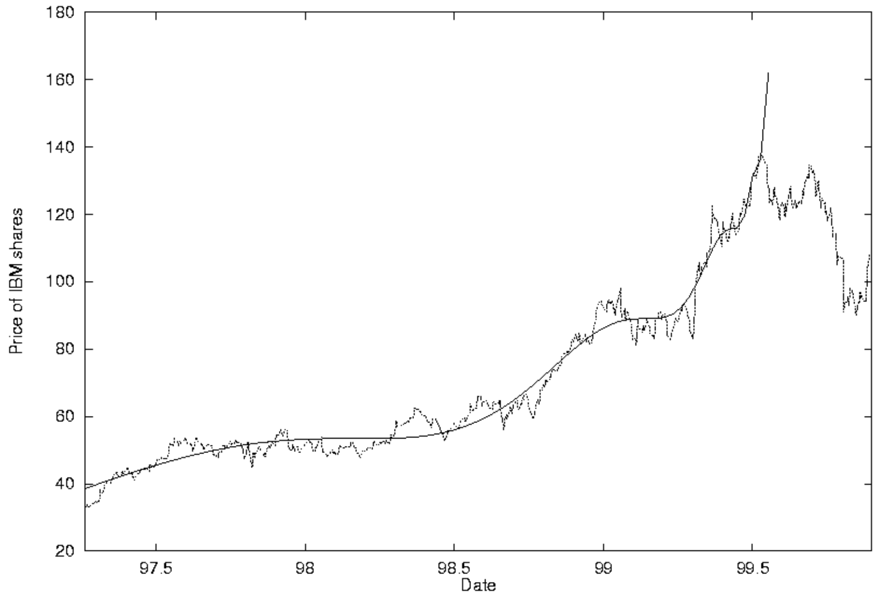

Figure 2: Best r.m.s fit with Eqn. (14) to the price of IBM shares.

3.1.4 A modification

A modified version of Equation (14) has been put forward [8] to study the precursory phenomena far away from critical point:

$$\log \frac{[p_c]}{p(t)} \approx \frac{(t_c-t)^\beta}{\sqrt{1+\left(\frac{t_c-t}{\Delta_t}\right)^{2\beta}}}\left[B_0+B_1\cos\left(\omega\log(t_c-t)+\frac{\Delta_\omega}{2\beta}\log\left(1+\left(\frac{t_c-t}{\Delta_t}\right)^{2\beta}\right)+\phi\right)\right] \quad (21)$$

where we have two new parameters: Δ_t and Δ_ω. The new effects which this equation includes are

- The power law tapers away far from the critical point.
- The log-frequency shifts from $\frac{\omega+\Delta_\omega}{2\pi}$ to $\frac{\omega}{2\pi}$ as we approach critical point.

Both these events take place over the same time scale Δ_t. Thus, we now allow for a pre-critical regime where prices oscillate around a constant level with a different log-periodicity.

In figure 3, we fit the logarithm of the Dow Jones Industrial Average index prior to the 1929 crash thus extending the fitting region from 2 to almost 8 years. The relative error to this fit has been found to be less than 10% on the entire time interval.

3.1.5 The next crash?

The results of the above discussion do not justify that stock market crashes can be exactly predicted ahead of time [9]. For one thing, the crash may be a one-day process or it may take months to occur, so there is no definite time of crash. Also, one needs to know

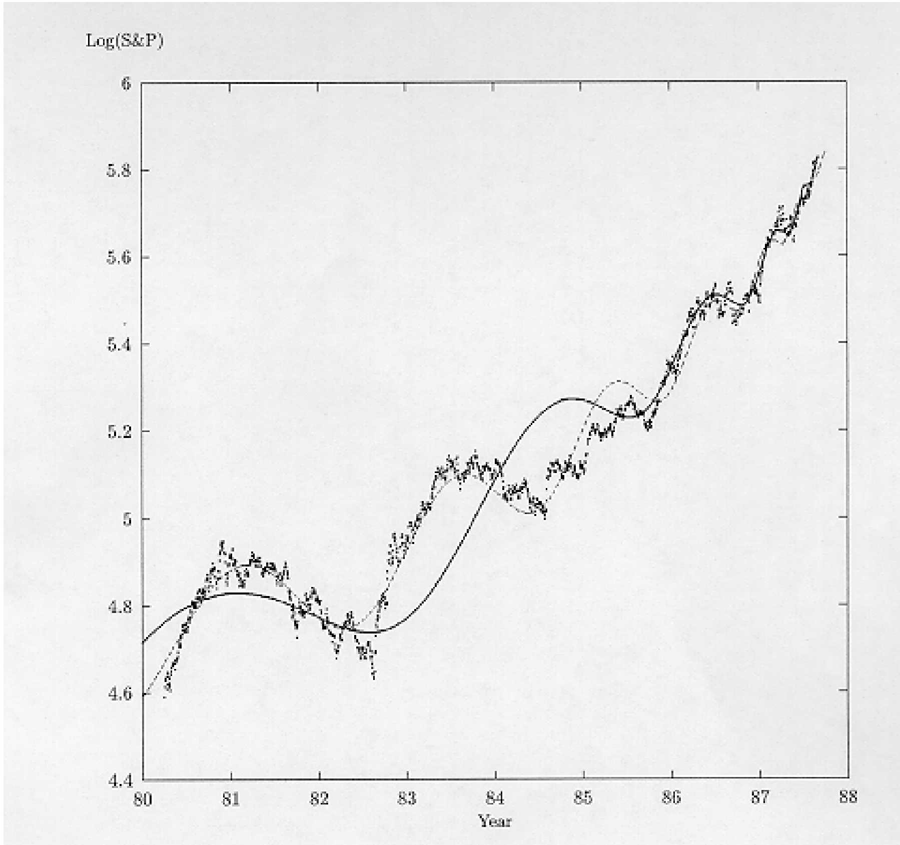

Figure 3: Time dependence of the logarithm of the NYSE S& P 500 from Jan 1980 to Sep 1987 best fit by Eqn.(21) (thin line). The thick line is the best fit by Eqn. (14) on the subinterval from July 1985 to the end of 1987, and is shown on the entire interval.

when to start looking for log-periodic structures, which involves improving our first-order approximations. One has to be careful of sampling enough data, otherwise accidental log-periodic corrections may emerge. Moreover, the probability that there may be no crash leads to false alarms, and researchers are trying to extend the methodology to improve reliability of predictions. It should be kept in mind that if predictions were accurate, traders could act on them , and crashes would never happen! So there is probably some information hidden in the stock market which we have not deciphered (and maybe will not be able to), leading to macroscopic behaviour which is not understood at all on the microscopic level.

3.1.6 The Growth Era

Another recent discovery [10] has been of that relating the world population to the economy of the United States. The increase in population is causing people to worry about the sustainability of global ecosystems, weather systems, etc. Others believe that the problems associated with an increasing growth rate will be taken care of by innovations like the Internet revolution. Noting that the population growth is intrinsically linked to its carrying capacity, we quantify the carrying capacity by a major financial index. On comparing the Dow Jones Industrial Average and the human population over two centuries, it has been found that *both* systems are exhibiting log-periodic oscillations towards a prediction of a singularity in 2058 ± 5 AD. Thus this growth contains its own limit, in the form of a singularity, which is being interpreted as an irreversible change to a qualitatively new behaviour. However, the transition may not be as abrupt as it looks, because the recent slowing down of the population rate will probably smooth it out.

3.2 Earthquakes

A large and important effort is being carried out world-wide in the hope that, in the future, it may be possible to predict earthquakes. However, there are recent opinions that earthquakes may be inherently unpredictable [11]. This is largely due to the fact that theories proposed in the past have not delivered reliable and accurate results. Seismicity is characterized by a very rich phenomenology and variability which makes it very difficult to form a coherent framework for prediction and explanation. There are a number of factors-chemical, and mechanical- that have to be integrated into a good model. The role of heterogeneities (like groundwater) is still not fully understood. As a result, there is currently no accepted model for earthquakes. Even though most models are closely related in their microscopic dynamics, the collective phenomena exhibited are very distinct.

The approaches to earthquake forecasting are to either look for precursory phenomena associated directly with the fault instability, or to look for more spatially extended changes [12]. The first approach has not been very reliable because of the absence of local precursors in many cases. However, quite a few large earthquakes have been preceded by an increase in the number of intermediate-sized events. So it is believed that seismicity is a problem involving the collective behaviour of a spatially extended fault network. These long-range spatial correlations have led some researchers to propose a self-organized model of the crust. However, earthquake data is scarce, and this model is still under debate.

3.2.1 Self-Organized Criticality

SOC refers to the spontaneous organization of a system driven from outside in a dynamical stationary state, which is characterized by self-similar distribution of event sizes and fractal geometric properties. Here we use the term SOC as defined in [13] , i.e., for the phenomena in continuously driven out-of-equilibrium systems made up of many interactive components, possessing the following properties -

1. Highly nonlinear behaviour, or, essentially a threshold response.
2. A very slow driving rate.
3. A globally stationary regime.
4. Fractal geometric properties and power distribution of event sizes.

The crust obeys these four conditions:

1. The stick-slip friction provides the threshold response. We can also compare the threshold to some other variable which characterizes the behaviour of the fault on increasing applied stress.

2. The slow driving rate is that of the slow tectonic deformations which are exerted at the borders of a plate by its neighbouring plates, and at its base by the lower crust and mantle.

   The presence of the threshold causes the system to slowly accumulate the slowly increasing stress until an instability is reached, while the slow driving allows for a response which is decoupled from the driving itself.

3. The stationarity condition ensures that the system is not in a transient phase. This helps to distinguish a long-term faulting from, say, a laboratory rupture experiment.

4. The power laws and fractal nature bring the scale invariance into the picture.

To get these conditions at a microscopic level, we have a few models, the most common being the Burridge-Knopoff model, which consists of a system of blocks and springs pulled slowly across a rough surface, with periodic boundaries [14].

3.2.2 Predictions

This section refers to the work done in [12]. The quantity which is used to predict earthquakes is called the cumulative "Benioff" strain, which is defined as

$$\epsilon(t) = \sum_{n=1}^{N(t)} E_n^q \tag{22}$$

where $q = \frac{1}{2}$ and E_n is the energy release during the nth earthquake during the time of observation up to the current time t.

We fit Eqn. (14) to the cumulative Benioff strain release for magnitude 5 and greater earthquakes in Northern California for the period 1927-1988 (Figure 4). The prediction of the earthquake was $t_f = 1989.9 \pm 0.8$ for the Loma Prieta earthquake, which actually occurred on October 18,1989 i.e. 1989.8.

3.2.3 Discussion

In earthquake analysis, we need to be very cautious because of the lack of a large enough data set. Moreover, The noise in the data may introduce log-periodic behaviour where none exists [15]. It is still to be understood what the physical mechanism is that introduces DSI. The SOC model used is subject to debate, because there might be an external parameter which has not been considered which actually has to be tuned so that the system lies in its stationary regime. Other questions relate to the suitability of the Benioff strain as the relevant variable and the value of q.

4 Conclusion

In this paper, we discussed the applications of log-periodic scaling to predict earthquakes and financial crashes. There are a large number of systems that have not been discussed here, and many more that have not even been studied. Using log-periodic scaling, it should be possible to predict the singularities of any system with a hierarchical structure,

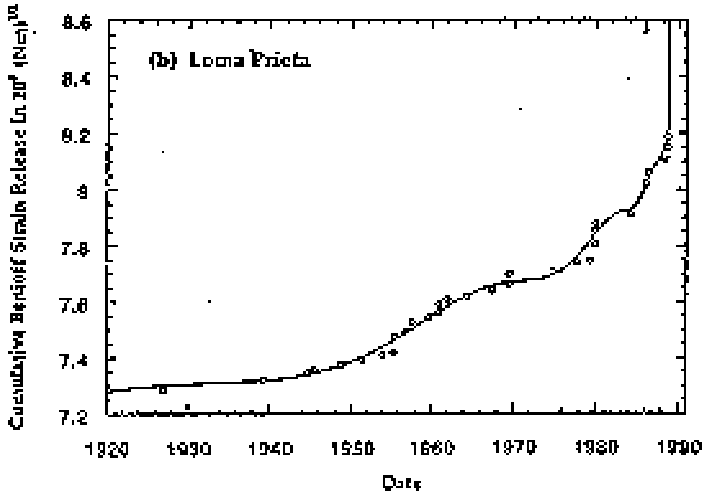

Figure 4: Cumulative Benioff strain released by magnitude 5 and greater earthquakes in the San Fransisco Bay area prior to the Loma Prieta earthquake fit to Eqn. (14).

but this leaves us with lots of uncharted territory, because not only do we have systems which have built-in hierarchy in their geometry, but also systems which break their continuous invariance and become discretely scale-invariant! Moreover, the predictive methods, though fairly good, have not yet attained the reliability that is needed of them, especially for earthquakes. These predictions are heavily data-dependent, and therefore, suffer from the same constraints as those that the people collecting data have. These results are only as good as their underlying models are, and research is actively being carried out to come up with intuitively appealing models which could be used reliably to predict and which could coherently explain our results with surety.

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