Critical Behaviour at the Onset of Black Hole formation

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Abstract

Critical phenomena found in the formation of black holes presents a very rich phenomenology from a relatively simple system. Scaling, universality, power laws, type I and type II phase transitions are found, showing that there is no limit for the applicability of these concepts. This paper presents a review of the topic which is expected to be understandable to people with no prior knowledge of general relativity or quantum field theory. A suitable collection of references is offered for those wanting to learn more about the topics covered.

1 Introduction and Outline

1.1 Introduction

For an isolated system in general relativity there are tipically three outcomes: it either collapses to a black hole, forms a stable star or explodes and disperses leaving empty space behind. Consider an (infinitely dimesional) manifold, each of its points representing one of the possible initial conditions of the system. Let's call this manifold the phase space. Some of the points (or initial conditions) in the phase space will produce a black hole and some others will end up as empty space. The multidimensional "line" that separates those two kinds of points divides this phase space into two regions with different qualitative behaviour, that is two phases. Thus, this line represents itself a phase transition. Near this phase transition the phenomena of universality, scaling and scaling laws have been observed in numerical simulations as well as equivalents to type I and type II transitions.

Several different systems have shown this behaviour, so it seems to hold quite generally for generic systems.

The topic, which has been said to be the most important recent contribution to general relativity, is an area of very active research and has interesting connections with important classical problems like cosmic censorship, mass distribution of primordial black holes and quantum gravity.

1.2 Plan of this paper

I begin by giving a description of the system in which criticality was first observed by Choptuik [1] . Concepts of general relativity and field theory useful to understand the topic are explained in that part, too.

The next section describes how the phenomena encountered changes with different systems that have been tried later on. This includes important extensions to systems with no spherical symmetry or with charge different from zero.

Section 4 offers a renormalization flow picture of what's going on and an explanation to the universality and power law observed.

The last section is devoted to the implications of critical phenomena in other parts of general relativity such as cosmic censorship, primordial black holes and quantum gravity

2 Basic scenario

The phenomena of criticality in black hole formation was first observed by Choptuik [1] in a numerical simulation of a scalar field minimally coupled to general relativity. Let me explain what this means.

2.1 Basic concepts of general relativity

One of the main protagonists of general relativity is the metric. The metric is what gives the distance in space-time between two points (events). In special relativity this metric assumes the familiar form:

$$\underbrace{ds^2}_{\text{proper distance}} = \underbrace{-dt^2}_{\text{time coordinate}} + \underbrace{dx^2 + dy^2 + dz^2}_{\text{space coordinates}} \tag{1}$$

Where the coefficients for the time and spatial coordinates are constants (-1 and 1 resp.). In general relativity these coefficients are allowed to change in time and space. A classical example is the Schwarzschild metric (expressed in *polar coordinates*, since it represents a system with spherical symmetry):

$$ds^2 = \underbrace{-(1 - 2M/r)}_{\text{this is not -1 anymore}} dt^2 + \underbrace{\frac{1}{1 - 2M/r}}_{\text{this is not 1 anymore}} dr^2 + \underbrace{r^2(d\theta^2 + sin^2\theta d\varphi^2)}_{\text{this part is the same as usual}} \tag{2}$$

The fact that these coefficients are not -1 and 1 as before reflects that the spacetime has **curvature** as opposed to the spacetime represented by equation (1), which is referred to as **flat spacetime**. This curvature (represented by the form of the metric) will then determine the movement of the particles in the spacetime. On the other hand, these particles carry some energy (mass, kinetic...etc). This energy will be the source for the metric: the distribution of mass-energy will determine the form of the metric (and the curvature). In summary: mass-energy generates curvature in spacetime, this curvature tells mass-energy how to move and this movement will change the distribution of mass-energy, hence changing the curvature and initiating the process again. The metric stores all the information on the curvature. Such is the dynamics of general relativity. As a last technical detail let me remark that in the mass-energy gravitational energy is not included, all the gravity effects are taken into account in the curvature.

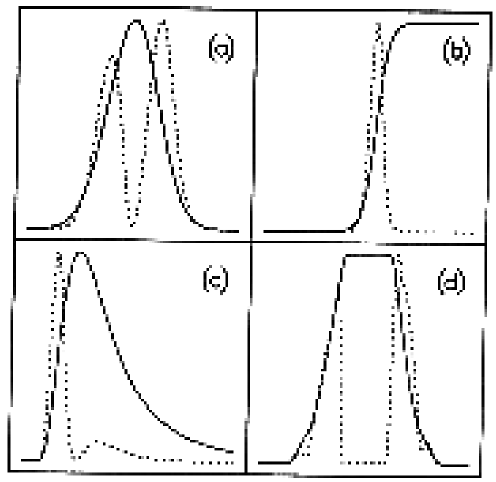

Figure 1: These are the typical initial profiles for the scalar field (solid line) and the mass-energy density (dotted lines) for different families of data as a function of the radius. Note that for this field the mass-energy density is given by the time derivative of the field. Taken from [1]

Everything that was said above for particles is true for **fields**. A field assigns a quantity (number, vector ... etc) to a point in space time. A familiar example are the electromagnetic fields $\vec{E}$ and $\vec{B}$. To each point in space and time a vector is assigned: $\vec{E}$(t,x), $\vec{B}$(t,x). Also, we know that to each point we can assign an energy density given by:

$$U(t,\vec{x}) = \frac{1}{2}(\vec{E}^2(t,\vec{x}) + \vec{B}^2(t,\vec{x})) \tag{3}$$

In the same way as stated for particles this energy will generate curvature, and this curvature will determine how this fields will change in time. Vector fields are not the only kind of fields that we may find. There are also scalar fields (assigns a *scalar* to a point), complex fields (assigns a *complex number*) and many other weird ones. The important point is that for all of them there is an energy density and equations of movement coupled to gravity (i.e. given by the curvature). The reason fields are used is because all particles are represented by fields.

2.2 Universality, scaling law and scaling exponent

Now let me go back to the problem at hand. As said before, the phenomena of criticality was first observed by Choptuik [1] in a numerical simulation of a scalar field coupled to general relativity.

In the simulation, the initial conditions (see fig. 1) of the field were such that all the mass-energy of the field was concentrated in an spherical shell and traveling inwards (imploding). Several different initial profiles were used, each of them being a different family. For each of these families there is a parameter p characterizing the strength of the gravitational interaction. This could be the amplitude of the initial gaussian distribution of the field or the width of this gaussian... etc. Each family has a value of the parameter p_{weak} such that in the limit p $\rightarrow$ p_{weak} the final spacetime is flat and the scalar wave packet implodes through the center of the shell and then escapes to infinity. On the other hand, there is another value of the parameter p_{strong} such that in the limit p $\rightarrow$ p_{strong} the gravitational attraction is strong enough to bind part of the mass into a black hole in the center of the initial shell. Between these two values there is a value p^* where black hole formation happens for the first time. The region in which $p > p^*$ is called *supercritical* and the one in which $p < p^*$ is called *subcritical*.

What Choptuik found is that for all the different families when p was very close to p^* the variables that characterize the system, such as the coefficients of the metric or the value of the scalar field, all have the same behaviour at an intermediate time of the simulation. That is, after some time after the initiation of the simulation and before the field either collapsed into a black hole or dispersed into infinity, the system had an **universal** behaviour for all families. This can be seen in fig. 2.

Furthermore, let Z any of the variable that characterize the system and express it in terms of logarithmic variables of r and t (remember that this is a problem with spherical symmetry) in the following way:

$$\begin{aligned} \rho &= ln(kr) \\ \tau &= ln(k(T_o^* - T_o)) \end{aligned} \tag{4}$$

Choptuik found a remarkable **scaling** relation for solutions with p near the critical value p^* :

$$Z^*(\rho - \Delta, \tau - \Delta) = Z^*(\rho, \tau) \tag{5}$$

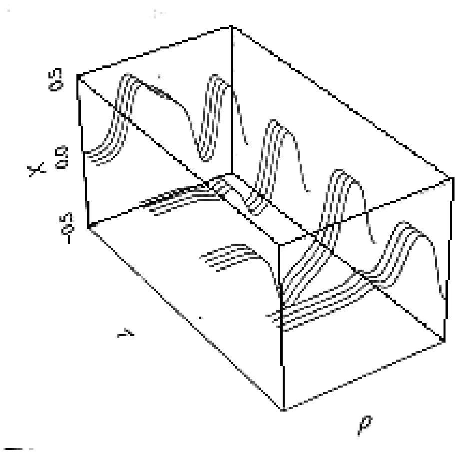

Figure 2: Here we can see the universal behaviour. Each group of four lines is the profile for four different families undergoing near-critical evolution. As we can see all families evolve in the same way. X is one of the field variables, τ and ρ are related to time and radius (see text). Taken from [1].

Where k and T_o are properly defined constants. Z^* denotes that p is close to p^* and, consequently, is the same for all families (universal behaviour). Δ is thus a *universal* constant with a numerical value of 3.4 . Paraphrasing Choptuik in [1], what this means is that if I freeze a critical evolution at some time where the universal behaviour is taking place and examine the profile of Z out to a maximum radius, then continue the evolution for a certain time δT_o and reexamine the solution on a scale $e^\Delta \approx 30$ times smaller than previously, I will see the same profiles. If I then wait and additional time interval $\delta T_o / e^\Delta$ and "zoom in" by another factor of e^Δ, I will again see the same profile.

Finally, in the supercritical regime (is $p > p^*$) the mass of the black hole is described by following **power law**:

$$M_{BH} \simeq c_f (p - p^*)^\gamma \tag{6}$$

Where c_f is a constant dependent on the family, but γ is a **universal scaling exponent** for all families, with a value of $\gamma \approx 0.37$.

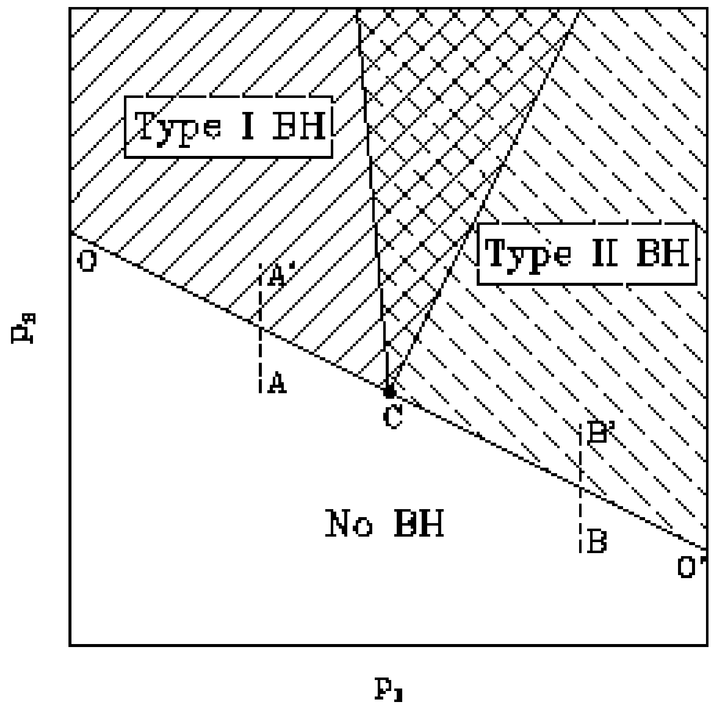

Figure 3: This is the phase diagram for a Yang-Mills field, showing type I (discontinuous jump in mass along phase transition across AA') and type II (continuous change of mass along phase transition across BB'. Taken from [2].

3 Extensions of the basic scenario

The purpose of this section is to show that the criticality phenomena are not a special case inherent to a particular field or symmetry, but a rather general situation in the dynamics of general relativity. Only a few examples will be given.

3.1 The Yang-Mills field

Criticality has also been observed in other fields as the Yang-Mills field [2]. In this particular case an analogy to type I and type II transitions can be found. Figure 3 is the phase diagram for this system. On the axis we have two parameters of a family which is considered to be sufficiently general. Starting at point A and finding the solutions for the different values of p up to the point A', we find that below a certain value of p_2 no black hole is formed. Above that point a type I black hole forms. What this means is that across the phase boundary the mass of the black hole changes discontinously from being zero (no black hole) to a finite amount (dependent on the initial conditions), in contrast the case shown above in equation [6] where $M \to 0$ for $p \to p^*$. So, taking our order parameter to be this mass the transition is said to be of type I. On the other hand, in the trajectory going from B to B' the mass changes continously, according to equation [6]. This is then called a transition of type II. Universality and scaling are also seen in this case much as in the case of the scalar field. The main important difference is that in this case $\gamma = 0.2$, thus signaling that the critical exponent depends on the field chosen.

It seems to be the case (4.3 of [3]) that systems with a single length scale in the field equations are in the same universality class. That is, they have the same critical exponents, universal behaviour and scaling laws. A single length means that there is only one fundamental constant. In general relativity and quantum field theory all units are measured in length units through conversion factors. Systems with two length scales (a mass and the charge of the electron, for example) would then exhibit different critical exponents.

All these results are numerical.

3.2 Non spherically symmetric systems

Little has been done out of the spherically symmetric case because of its complexity and the great numerical resolution required. Nonetheless attempts on axisymmetric systems have been carried successfully for the scalar field [4],[5] finding a critical exponent $\gamma = 0.36$ equal to the one in the symmetric case:

$$M \propto (p - p^*)^\gamma \tag{7}$$

This is particularly important because it shows that scaling is not an artifact of spherical symmetry.

3.3 Black holes with charge and angular momentum

Analitical perturbative approaches have been tried for black holes with charge [6] and angular momentum [7] with a scalar fields. The results indicate critical phenomena with scaling laws for the charge and momentum of the black hole of the type:

$$Q \sim (p - p^*)^\delta \tag{8}$$
$$L \sim (p - p^*)^\mu \tag{9}$$

With the exponents predicted to be $\delta = 0.88$ and $\mu = 0.76$. The first result has been confirmed by computer simulations. Note that in spite of having a scalar field, now there are other mass scales in play (the value of the quantum of charge in the first case, for example).

4 Renormalization group flow picture

It has been noted [3] that the time evolution of these kind of systems is similar to the renormalization group flow in the space of initial data. Instead of a space whose axis are the coupling constants, we will have a space in which each point represents some initial conditions. To each point a certain form of the metric and state of the field is assigned. This will be our phase space. There are some gaps with this definition, but they won't hinder our qualitative understanding.

Now, in this space there are three basins of attraction. Isolated systems in general relativity usually end up collapsing to a black hole forming a star or dispersing to infinity. These will be our basins of attraction (or fixed points). For our current case we will only consider the basins corresponding to either black holes or total dispersion. The boundary that separates systems that will end up in any of these two different fixed points is called a critical surface. By definition, all points starting in this critical surface will never leave it. Let's imagine that there is an attracting fixed point in this critical surface. This seems to be the case for the systems treated above. All points in the critical surface will end up in this critical point. Let's now imagine a point starting close to the critical surface, but not quite in it. At the beginning it will follow a flow very close to the one for a point at the critical surface, it will get nearby the critical point and stay there for a while, finally ending up

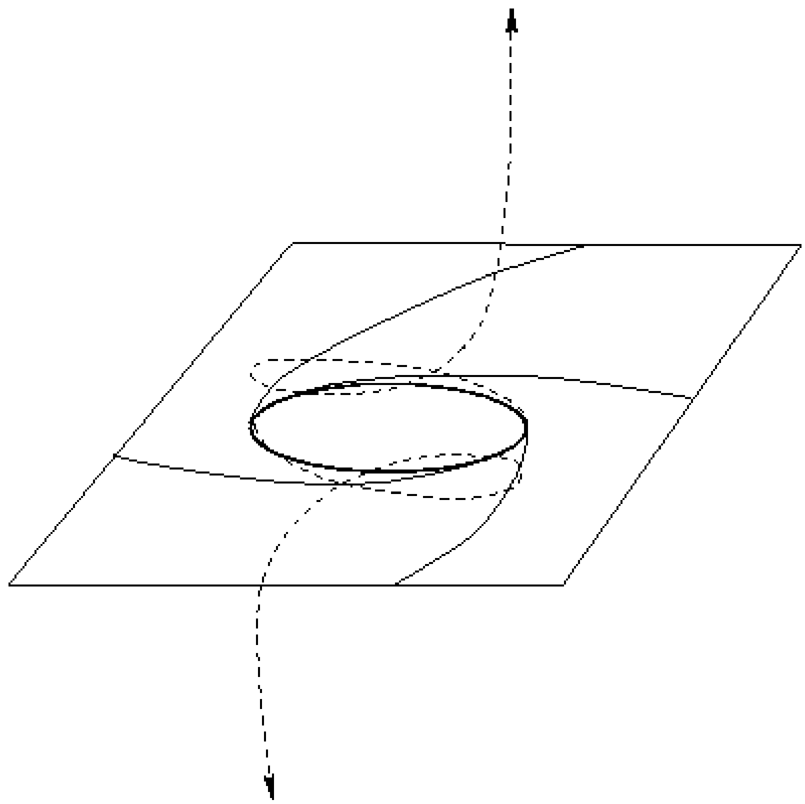

Figure 4: The phase space picture for criticality phenomena. The plane represents the critical surface. The circle (fat unbroken line) is the limit cycle representing the critical solution. The thin unbroken curves are spacetimes(systems0 attracted to it. The dashed curves are spacetimes repelled from it. There are two families of such curves, one forming a black hole, the other dispersing to infinity. Taken from [3]

in one of the basins of attraction (black hole of flat space-time). This is the origin of the universality:" any point close to the black hole threshold (on either side) evolves to a space-time that aproximates the critical space-time for some time. When it finally approaches either empty space of black hole it does so on a trajectory that appear to be coming from the critical point itself. All near critical solutions are passing through one of these two funnels. All details of the initial data have been forgotten, except for the distance from the black hole threshold." [3] See fig. 4 for a phase space picture.

Critical exponents can be calculated in a similar way as the divergence for the correlation length ξ in renormalization group theory (3.3 of [3]).

5 Implications of criticality

5.1 Naked singularities and cosmic censorship

The most direct consequence of Choptuik's results lies in the area of naked singularities. Let me first explain what this is. A singularity is a point of spacetime in which the curvature diverges. As an example, let's have a look at the Schwarzschild spacetime metric again:

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{1}{1 - 2M/r}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (10)$$

We see that we have a singularity in the metric for $r = 0$ (we have another one for $r = 2M$, but this turns out to be a singularity of the coordinate system only) and that is a curvature singularity, too. This singularity, though, is covered by an event horizon. The schwarzschild metric has an event horizon at $r = 2M$ (this can be deduced from the metric). An event horizon is a surface such that nothing inside could ever reach the outside of the event horizon. Whatever is inside it is effectively disconnected from the rest of the universe.

Now, in every spacetime known previously whenever there was a singularity there was always an event horizon surrounding it. This lead to the question whether this would always be the case. That is, whether there is a cosmic censor dressing up all indecent naked singularities with an event horizon. This is the topic of cosmic censorship.

The critical spacetime found by Choptuik certainly presents a singularity with no event horizon. So that leads to the consequence that cosmic censorship can only be true in a milder formulation: "*generic* smooth initial data for reasonable matter do no form naked singularities."

The difference from previous formulations resides in the word "generic" since the critical spacetime is a very specific point in the manifold; it doesn't even encompass a set of points.

5.2 Astrophysical implications

The problem with real world applications of critical phenomena is that a very fine tuning is required. To get a spacetime affected by the universality we would require to have the parameters of the system very close to the critical point. We don't really have to care about the fact that some of the model exposed here (shells of mass imploding inwards and such) do not look realistic, since these phenomena seem to be quite general in formation of black holes.

In any event, Niemeyer and Jedamzik [8], came up with a scenario in which this fine tuning can happen. In the early universe, quantum fluctuations of matter and the metric can be important, forming massive stellar objects, like galaxies. If this fluctuations are large enough, they can collapse immediately, giving birth to what are known as primordial black holes. Large quantum fluctuations are exponentially more than smaller ones $P(\delta) \sim e^{-\delta^2}$, with δ being the density contrast of the fluctuations. For fluctuations smaller than a certain critical δ_{crit} there is no creation of black holes whereas for bigger fluctuations the probability of their existence decreases very rapidly. This gives a sharply peaked probability of the spectrum of black holes near δ_{crit}. So for black holes near the limit where formation is possible we have a peaked distribution. This is the required fine tuning. Assuming these black holes to be of type II their mass would be pretty small because of [6] (p very close to p^* for these black holes) and that would mean that the peak of the spectrum of primordial black hole masses may be at smaller values of the black hole mass than expected.

No information has been found on the relevance of critical phenomena at present time. It seems unlikely that the necessary fine tuning actually happens.

5.3 Quantum gravitation

From equation [6] we can see that the mass of the created black holes can be as small as desired by making p as close as p^* as wanted. This could lead to the creation of black holes in the scale of Planck's length, where quantum effects are important. Unfortunately no quantum theory of gravity is yet available and most attempts had to limit to the semiclassical limit. Most of the results found are quite inconclusive. See 5.5 of [3] for more details.

6 Conclusion

Critical collapse has evolved into an exciting subfield of general relativity in the few years since the publication of the first paper (1992). Besides having interesting connections with other areas of general relativity, it has an intrinsic interest for the richness of the phenomenology involved. Most of the work done so far has been numerical, and there are few analitical results.

Further development will come from the development of analitical understanding of the phenomena, more rigorous definition of concepts (like phase space), and simulations including cases outside the spherical symmetry.

Finally, critical phenomena is (arguably) the most important contribution from numerical relativity, and will probably continue to produce interesting results and take the numerical relativity to its limits.

7 Thanks

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