Wall Bounded Turbulent Shear Flow: Has the *Law of the Wall* Been Broken?

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Abstract

Though the Navier-Stokes equations for fluid flow have been known for over a century, unstable solutions, known as turbulent flows, are still not well understood. For some very simple classes of fluid flow, however, simple dimensional arguments combined with physically-motivated assumptions have led to relations for some average properties of the flow. In the 1930's a logarithmic form for the average velocity profile of wall-bounded shear flow proposed, based on one such argument. Accepted for over six decades, this form, known as the "Law of the Wall", has come into question as a result of the work of Barenblatt, Chorin, and others, who contend that the assumptions leading to this law are incorrect and must be replaced, yielding a power-law scaling for flows in the limit of low viscosity and high velocity. A introduction to the concept of turbulence will be given, followed by a brief history of the "Law of the Wall". Experimental data which confirms the breakdown of the "Law" will be given along with the new theory proposed by Barenblatt and Chorin (B&C). Critical response to the B&C theory will be considered, followed by a brief discussion of the broader implications of this work.

1 Introduction

Throughout the latter half of the nineteenth century and the first half of the twentieth century, the field of fluid mechanics was brought from its infancy to a state very near maturity. With the development of the Navier-Stokes equations, it was understood that the broad range of phenomenology encountered in the field could be deduced, at least in principle, from a single set of coupled partial differential equations governing the flow field. With the advent of high-speed digital computers in the past few decades, these equations coupled with algorithms and hardware to solve them provided a quite general method to quantitatively understand even complicated flows. It would seem that the field has reached full maturity, and indeed it has in many respects, yet with one major exception – turbulence.

There exists, for every possible geometry and flow parameters for a system, a steady-state solution to the Navier-Stokes equations. These solutions, however, will not necessarily express the actual behavior in a physical implementation of the system. In order for a such a steady-state solution to be reflected in Nature, it must be stable under small perturbations to the system parameters. That is to say, the effect of any small perturbation must decay exponentially with time. Those systems whose steady-state solutions are not stable, i.e those in which a small perturbation grows in time, exhibit a chaotic behavior known as turbulence. Though nearly a century has past since turbulence was first identified, it remains a topic which is fundamentally not well understood and is currently an area of active and vigorous research.

Though the microscopic processes involved in turbulence are still unclear, some progress has been made. Despite the chaotic nature of turbulent flow, it has been possible to understand certain properties of the flow when averaged over a time period which is long in comparison to the time scale of the turbulent fluctuations. For example, computational codes intended to model properties of flows with turbulent regions have done so with large success by replacing the true viscosity of the system by an empirically-determined effective viscosity which accounts for additional momentum and energy transfer by the convective fluctuations involved in turbulence.

The central concern of this paper, however, is on the analytic side of developments. While we currently lack the ingenuity to analytically extract a full understanding of turbulent behavior directly from the Navier-Stokes equations, in certain limiting regimes it is possible to make dimensional and scaling arguments to describe the behavior of many long-time average properties. One such argument was first made in the 1930's regarding the average velocity profile for turbulent flow over an infinite flat plane. Accepted for nearly seven decades, the assumptions and results of this argument have recently come into question due to the work of Barenblatt, Chorin, and collaborators (henceforth BCC). This specific example is the central concern of this letter.

2 Wall-Bounded Turbulent Shear Flow

For the remainder of this discussion, we will concern ourselves with a simple physical system: tangential fluid flow bounded by a wall. Perhaps the simplest flow of this type is that over an infinite flat plane. We would like to make contact with experimental data, however, and as infinite planes are hard to come by in the laboratory (though finite approximations thereof are utilized in experiment), we consider instead flow through a cylindrical pipe. Note that the arguments presented henceforth are not restricted, however, to that particular geometry, but may be generalized with a bit of effort.

Before beginning in our analysis of the physical system, it will become necessary to introduce a bit of terminology. Consider a cylindrical pipe of diameter d whose axis coincides with the x-axis. The time average velocity, which we term u, will be in the positive x direction, though the instantaneous velocity will, in general, have transverse components. Note that, henceforth, any mention of velocity should be interpreted as the time average of that quantity unless

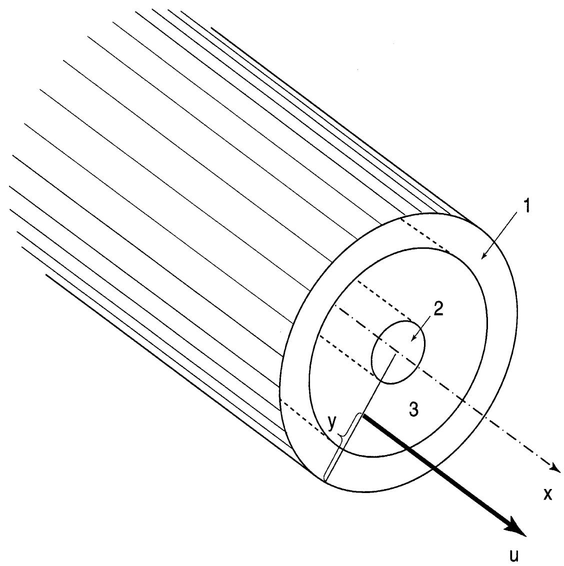

Figure 1: Schematic of pipe flow delineating flow regions. Borrowed from [2]

otherwise noted. Let the y represent the normal distance from the pipe wall. In general, u will vary as a function of y. We adopt the standard "no-slip" boundary condition that the velocity vanishes at $y = 0$. In the limit of highly turbulent flows, it is possible to characterize the flow into three general regions:

- 1) The viscous sublayer is the region very near the wall. In this region, the velocity gradient, $\partial u/\partial y$ is very large so that the momentum transfer due to the Brownian motion of the fluid molecules is comparable to that due the turbulent vortex motion.

- 2) The the axial region is that near the center of the pipe. Considering the cylindrical symmetry, the velocity gradient must vanish along the central axis if we rule out the possibility of a singular momentum source or sink along this line. Thus, the velocity is relatively constant in this region;

- 3) The intermediate region is that in between 1) and 2), where the velocity gradient, $\partial u/\partial y$, is relatively small compared to that in region 1), but non-zero. It is region 3) that is of interest in this discussion.

In fluid dynamics, it is often profitable to introduce dimensionless versions the physical parameters. In many circumstances, it is possible that physically distinct systems, when expressed in dimensional form, become equivalent and have the same behavior when expressed in dimensionless form. This property is known as similarity, which has important implications in this discussion.

We proceed, then with our dimensionless variable definitions. We begin with the average velocity, $u(y)$. We define

$$\phi = \frac{u}{u_*}, \tag{1}$$

where u_* is termed the "friction" velocity and is defined by

$$u_* = \sqrt{\tau/\rho} \tag{2}$$
$$\tau = \frac{dp}{dx}\frac{d}{4}. \tag{3}$$

Here, dp/dx, represents the pressure drop per unit length of pipe, and d is the pipe diameter. We must define our non-dimensional distance from the wall,

$$\eta = \frac{u_* y}{\nu}, \tag{4}$$

where $\bar{u}$ is the velocity of the flow averaged over a cross-section of the pipe and ν is the kinematic viscosity, defined as the dynamic viscosity divided by the fluid density. Next, we introduce a a dimensionless quantity utilized ubiquitously in fluid dynamics – the Reynolds number, Re.

$$Re = \frac{\bar{u}d}{\nu}. \tag{5}$$

Generally speaking, the Reynolds number gives an indication of how turbulent the flow for a system will be, with larger numbers indicating more turbulent flow. Furthermore, dimensional analysis shows that if a system's linear dimension, velocity, and kinematic viscosity are simultaneously scaled, the non-dimensional velocity profile, $\phi(\eta)$, will remain unchanged in the systems share the same Reynolds number.

3 The Law of the Wall

We consider for a moment the velocity profile in each of the flow regions define above. In the viscous sublayer, which is quite thin for the high-Re flows considered here, the velocity changes extremely rapidly from zero at the wall boundary. Here, molecular viscosity plays a critical role in the dynamics, so that the role of turbulence cannot be easily separated. In the central region, the velocity profile is relatively constant and thus uninteresting. We therefore draw our attention to the intermediate region.

Since the 1930's two distinct functional forms for the velocity profile, $\phi(\eta)$ have been used by researches and engineers to describe the flow in this region. The first is a simple power-law scaling.

$$\phi = C\eta^{\alpha} \tag{6}$$

Here, C and α vary slightly with Re. This form was utilized primarily in the early years of research and engineering in turbulence. An alternative logarithmic form was first introduced by Prandtl [15] and von Kármán [16].

$$\phi = \frac{1}{\kappa} \ln \eta + B \tag{7}$$

Here, the constants κ and B were assumed to be fully "Universal" – in this case independent of Re. For nearly seven decades, this equation, known as the *Law of the Wall*, has been generally accepted and utilized with large success. There have been occasional suggestions of the requirement of Re-dependent corrections, but as yet this requirement has not been accepted by the community at large. We present here a simple dimensional argument, when combined with an assumption, leads to (7). Our presentation follows the refined argument of Landau and Lifshitz [6] rather than the original ones of Prandtl and von Kármán.

In this argument, we consider the gradient of the velocity, $\partial_y u$, rather that the velocity itself. The value of the velocity itself in the intermediate regions depends strongly on the functional form of the velocity profile in the viscous sublayer, where the functional form we derive will be invalid. The system can be fully parameterized by the values y, τ, d, ν, and ρ. Therefore, we have

$$\partial_y u = f(y, \tau, d, \nu, \rho, \bar{u}) \tag{8}$$

We continue our argument with dimensional analysis. For the convenience of the reader, we begin by listing the dimensions of each of our parameters. The function f in (8) must have dimensions T^{-1}. Note that f is a function of six variables, three of which have independent dimensions. Note also that $\bar{u}$ is the velocity averaged over the pipe, so that it is a derived variable entirely determined by u and d. The Buckingham π theorem then states that we may write (8) in the form,

Quantity	Dimension
y	L
τ	$ML^{-1}T^{-2}$
d	L
ν	L^2T^{-1}
ρ	ML^{-3}
$\bar{u}$	LT^{-1}

$$\frac{y}{u_*} \partial_y u = \Phi(\eta, Re), \tag{9}$$

where

$$Re = \frac{\bar{u} d}{\nu}, \quad \eta = \frac{u_* y}{\nu}. \tag{10}$$

This follows from the fact that we have five independent variables, three of which have independent dimension. We may then rewrite this equation entirely in terms of our dimensionless variables,

$$\partial_\eta \phi = \frac{1}{\eta} \Phi(\eta, Re) \tag{11}$$

Now we come to the central hypothesis of the von Kármán-Prandtl law. For the systems considered in this discussion, the length scale defined by the quantity ν/u_* is on the order of microns. Therefore, in the intermediate flow region, η is a very large number. Then for sufficiently high Reynolds number at an intermediate distance well outside the viscous sublayer, the arguments of Φ are both very large. The traditional argument, first put forth by Prandtl and von Kármán is that we may then replace $\Phi(\eta, Re)$ by the limiting value $\kappa^{-1} \equiv \Phi(\infty, \infty)$. This yields, upon integration,

$$\phi(\eta) = \frac{1}{\kappa} \ln \eta + B, \tag{12}$$

where κ and B are universal parameters whose values are typically in the range of 0.42 and 5.1, respectively.

The von Kármán-Prandtl-von hypothesis is tantamount to the assumption that Φ is analytic in Re^{-1} and η^{-1} when these quantities approach zero. There is no *a-priori* reason to assume that these conditions hold, and furthermore we shall see in later sections that the analysis of data by BCC suggest, in fact, that they do not. In the next section, we consider a replacement for the von Kármán-Prandtl-von hypothesis, which leads to a scaling, rather than logarithmic, form for the velocity profile.

4 Similarity and Incomplete Similarity

Consider a function, g of n dimensionally independent parameters, $x_1 \ldots x_n$ and one parameter, b, whose dimensionality can be constructed from those of $\{x_i\}$. Consider

$$a = g(x_1, x_2, \ldots x_n, b) \tag{13}$$

The dimensions of both a and b must be derivable from the dimensions of $\{x_i\}$. Then let

$$[a] = [x_1]^{\alpha_1}[x_2]^{\alpha_2} \ldots [x_n]^{\alpha_n} \tag{14}$$

$$[b] = [x_1]^{\beta_1}[x_2]^{\beta_2} \ldots [x_n]^{\beta_n} \tag{15}$$

We follow traditional dimensional analysis and define the non-dimensional quantities

$$\pi = \frac{a}{x_1^{\alpha_1} \ldots x_n^{\alpha_n}} \tag{16}$$

$$\theta = \frac{b}{x_1^{\beta_1} \ldots x_n^{\beta_n}} \tag{17}$$

Then, invoking the π theorem,

$$\pi = \Phi(\theta) \tag{18}$$

We consider the limit of Φ as its argument tends toward zero. If this limit exists, then for values of the parameter sufficiently small, we may write,

$$\pi = C, \tag{19}$$

where C is the constant limiting value of Φ. This case is known as *complete similarity* is the parameter θ. This property is certainly not true in general. An alternative is that, in this limit, the function Φ takes on a scaling form in its argument.

$$\pi = C\theta^{\gamma}. \tag{20}$$

In dimensional variables,

$$\begin{aligned} a &= C\theta^{\gamma} x_1^{\beta_1} \ldots x_n^{\beta_n} && (21) \\ &= C x_1^{\alpha_1 - \gamma\beta_1} \ldots x_n^{\alpha_n - \gamma\beta_n} b^{\gamma} && (22) \end{aligned}$$

In this case, the exponents of the $\{x_i\}$ depend on γ, which cannot be determined by dimensional analysis. Furthermore, the functional dependence on b has not disappeared. In this case, it is said that π has *incomplete similarity* in θ. The next section discusses how this mathematical aside connects with wall-bounded shear flow.

5 The BC Hypothesis

We are now in the position to recognize that the von Kámán-Prandtl hypothesis is tantamount to the assumption that in the functional form,

$$\partial_{\eta}\phi = \frac{1}{\eta}\Phi(\eta, Re), \tag{23}$$

for sufficiently large values of its arguments Φ is completely similar in both, which leads upon integration to the logarithmic “Law of the Wall”. Recent analysis of data by BCC suggests that the above assumption is not born out in experiment, (which will be reviewed in section 6) and hence must be abandoned. In its place, those authors suggest a form of incomplete similarity and η and no similarity whatsoever in Re, yielding an expression of the form

$$\Phi(\eta, Re) = A\eta^{\alpha}. \tag{24}$$

Here, the lack of similarity in Re allows A and α to have Reynolds number dependence. Integrating, we arrive at our revised velocity profile,

$$\phi = \frac{A(Re)}{\alpha(Re)}\eta^{\alpha(Re)} + \text{constant}. \tag{25}$$

It is tempting to appeal to the no-slip boundary condition at the wall to eliminate the constant of integration. This functional form, however, is not valid in the viscous sublayer near the wall, so that this appeal is unjustified. BCC do indeed drop the constant, but do so with the cognizance it constitutes an additional assumption.

Given the form (25), we are left the task of finding the forms of $A(Re)$ and $\alpha(Re)$. BCC argue that the paramters should be of the form

$$A(Re) = A_0 + A_1 \varepsilon(Re) \quad (26)$$
$$\alpha(Re) = \alpha_0 + \alpha_1 \varepsilon(Re). \quad (27)$$

This, at first glance, seems to be merely a symantic redifinition of terms. In this form, however, rather stringent restrictions may be placed on ε. First, the paramters α and A are known to have a weak dependence on the Reynolds number and this dependence decreases with increasing values of the Reynolds number. Therefore, ε should be small and tend toward zero as the Reynolds number approaches infinity. Nonetheless, there is a measurable and appreciable dependence on Re even for the largest values that have been physically realized in experiment. Therefore, ε must not fall off too fast, such as in a $1/Re$ dependence. These considerations combined with some motivations from the statistical theory of turbulence, have led BCC to the final form,

$$\phi(\eta, Re) = (C_0 \ln Re + C_1)\eta^{\left(\frac{\alpha_1}{\ln Re}\right)}. \quad (28)$$

Fit to the best available data, due to Nikuradze [17], the paramters are determined as

$$\phi(\eta, Re) = \left(\frac{\sqrt{3} + 5\alpha}{\alpha}\right)\eta^{\alpha}, \qquad \alpha = \frac{3}{2\ln Re}. \quad (29)$$

It is important to note that the scaling law, (29) was derived with no more rigor than the universal logorithmic law, (7). Both were derived with specific assumptions, neither of which were more general nor better motivated by physicsal considerations. Given the present inablility to go directly from the Navier-Stokes equations to a macroscopic theory of turbulence, the decision between forms must be determined by comparison with experiment. Of course possibilities other than those obtained from either complete or incomplete similarity arguments are not ruled out. We will see, however, that the BC scaling law is in quantitative agreement with a broad range of data and provides a more accurate description, at least, than the logarithmic *Law of the Wall.*

6 Comparison with Data

We begin by considering Figure 2, a plot of the data obtained in Princton's *Superpipe* experiment. This data was utilizing the best available techniques and equipment and, for sake of brevity, it will be the only data presented in this discussion. The abscissa reprsents the the logarithm of η and the ordinate represents ϕ. If the logarithmic law is correct, all the data should fall along a

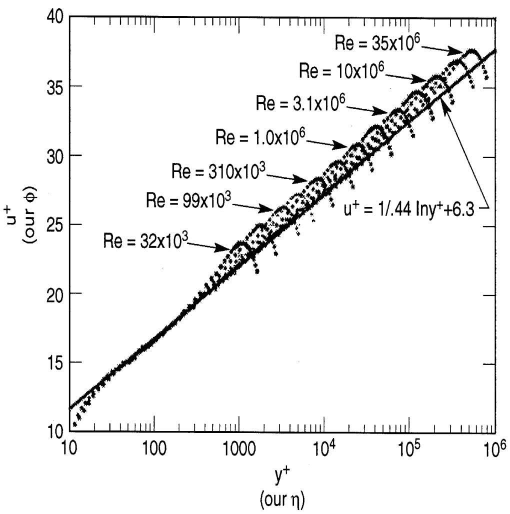

Figure 2: Data obtained from Princeton's *Superpipe* shows the Reynold's number dependent deviation from the universal logorithmic scaling law. [11] Note that despite the fact that this data is dated by seven decades, obtained under the guidance of Prandtl, it remains a milestone of quality data.

single line, independent of Reynolds number. Instead, we see a family of curves, distinct for each value of Re. Note that the each curve turns downward near the center of the pipe, outside the intermediate region where the arguments leading to both the logarithmic and scaling laws are valid. The question arises, then, how such a striking deviation from the universal logarithmic law could have been overlooked for seven decades. Some explanation may be gleaned from the schematic representation in Figure 3, borrowed from [2]. As the figure shows, each curve in the Reynolds number dependent family share an envelope which matches the logorithmic dependence. If lower-quality data is plotted with all data on a single plot, the Reynolds number dependence is not obvious. This is the case particularly if data is only available for the central part of the intermediate region, are truncated at too low a value of η.

Figure 4 shows the predicted curves based on the form (29). For $Re < 10^6$, the prediction is in good quantitative agreement with the data presented in Figure 2. For $Re > 10^6$, the theoretical curves slope upward a bit more rapidly than the experimental curves. BCC have attributed this discrepancy to roughness in the interior wall of the Princeton *Superpipe*. They argue that for $Re > 10^6$, the viscous sublayer thickness is sub-micron, so that any microscopic roughness of the wall may protrude into the intermediate region, changing the properties of the fluid flow. The roughness effect is apparently well documented (eg.[12]), and we will not comment on it further here.

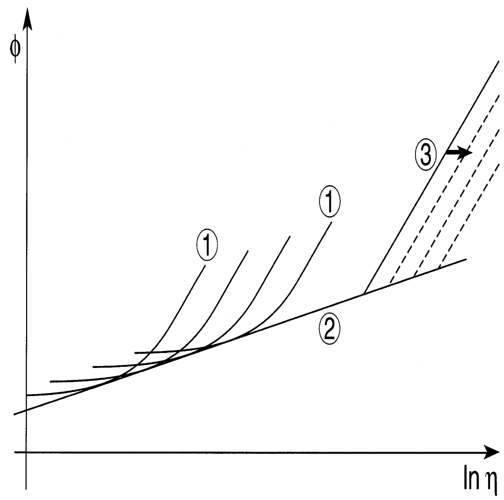

Figure 3: Schematic representation of how the Reynold's number dependence of the velocity profiles could have been overlooked. The family of Re-dependent curves share a common envelope which matches the logarithmic form. If data from only the central part of intermediate region is plotted, all of the points will fall very near this envelope.

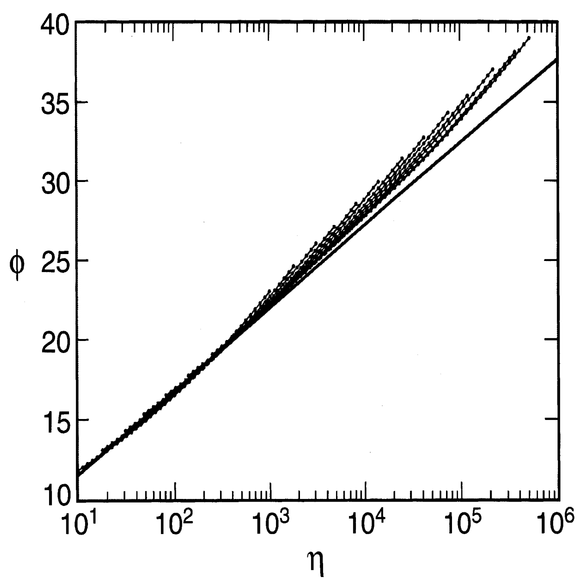

Figure 4: Above is plotted the predictions of the BC scaling law for the range of Re shown if Figure 2. The data is in quantitative agreement for $Re < 10^6$, but only qualitative agreement above this value.

7 Response to the BC theory

Since its introduction into the literature, the BCC theory has no supplanted the previouly excepted logorithmic law, nor has it been immune to criticism. Some authors have attempted to reaffirm the logorithmic law with experimental data. For example J. M. Österlund in his 1999 Stokholm thesis [13] presents a set of 70 mean velocity profiles in boundary layers with zero pressure gradient as evidence confirming the earlier *Re*-independent logarithmic law. In a recent response released on the LANL preprint server [14], BCP argue provide a convincing argument that the data was improperly processed by by the Stokholm group, leading to erroneous conclusions. The provide a description of a method for proper analysis and accompany their paper with seventy graphs showing the result of their analysis. They demonstrate furthermore that when properly processed, the large-*Re* datasets provided further evidence of *Re*-dependent scaling behavior.

Other authors have continued analytic work within the framework of complete-similarity without reference (and perhaps knowledge) of the recent work of BCP. In September of 1999, Chattopadhyay and Bhattacharjee of Calcutta submitted an article [10] to the LANL preprint archive which begins with the von-K'arm'an-Prandtl law and uses a randomly stirred model of turbulence to arrive at a universal value for the parameter κ. This article contains no reference to the work of BCP or, for that matter, any work more recent than 1993, aside from articles by the authors themselves. It is possible that the new form put forth by BCP was simply unknown to the authors.

Despite the theoretical and practical implications of a modification to the *Law of the Wall*, the attempts to put down the work of BCP. Although the physical setup of flow through a pipe seems simple, the experimental apparatus required for the capture of quality data is extremely sophisticated and expensive, and indeed few are in existence. Nonetheless, BCP have analyzed a large number of independently-collected datasets, all of which have given stronger qualitative and quantitative agreement with their proposed scaling law than the logarithmic law. While the BCP form may not be the last word in high-*Re* intermediate wall-bounded turbulent shear flow, they have supplied ample word that it is at least a later word than the von K'arm'an-Prandtl theory.

8 Broader Context

While the exact form of the *Law of the Wall* has important practical ramnifications, it has also significant theoretical implications beyond those mentioned above. Throughout this discussion, our arguments have focused on mathematical considerations and appeals to data, with little reference to the underlying physics of turbulence. We began with dimensional analysis and then considered two alternative assumptions leading to different final forms for the average velocity profile. These assumptions, however, do make contact, at least qualitatively, with the underlying physics. Here we briefly comment on these connections.

The velocity fluctuations that arise in turbulent flow appear at different length scales. This length scale is defined as the average distance over which the instantaneous velocity varies appreciably from the time-averaged value. The fluctuations at the largest length scale are the first to appear as Reynolds number is increased from the stable region to the unstable region. In general, the large-length-scale fluctuations have the largest amplitude and dominate the flow. As Reynolds number is increased, shorter length fluctuations enter. At some sufficiently high Reynolds number, it is believed that the flow reaches a point in which fluctuations at a minimum length scale, λ_0, enter. Flow at and above this Reynolds number are referred to as *fully-developed*.

Despite the amplitude of the fluctuations, the instantaneous velocity gradient in the large-length-scale fluctuation is negligible in comparison that at the shortest length-scale. Therefore, almost no energy is dissipated in the long length-scales. Fully-developed turbulence may then be considered a process in which energy is carried from long length-scales to shorter ones, down to the shortest ones, where the energy is finally dissipated. At the largest length-scales, which dominate the behavior of the flow, the viscosity is therefore unimportant outside the viscous sublayer where the gradient of the average velocity is large.

It is this physical argument that motivated the complete similarity assumption which led to the logarithmic velocity profile. Once the flow is fully developed, changes to the viscosity (and hence to Re) should not affect the large-scale behavior, such as the velocity profile in the intermediate region. BCP have demonstrated that empirical data force us to abandon the logarithmic law, the arguments which led to it must be abandoned, or at the very least modified. The question arises, then, as to whether *fully-developed* turbulence exists. The possibility exists that new length-scales are continually coming into play or that these length scales play an increasing role as Reynolds number is increased. An explanation of these issues involves a detailed disussion of stucture functions and is beyond the intended scope of the author. The interested reader may find detailed discusions in [3] and [2]. The perhaps erroneous arguments above are detailed in [6].

Thus, in addition to giving a more accurate qualitative and quantitative description of a restricted class of phenomena, the recent investigations of wall-bounded turbulent shear flows have given new insight to an as-yet relatively poorly-understood field.

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