Jamming Transition

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Abstract

Jamming phenomena arise in many systems. Granular materials, glasses and colloidal systems are just a few examples of it. This paper summerizes current effort of unifying these phenomena into a single jamming phase transition. Two examples, granular materials and colloidal suspensions, are discussed in detail to illusrate the idea. The strengths and weaknesses of this picture are addressed at the end of the paper.

1 Introduction

Glass forms when a liquid is quenched into a solid phase[1]. Granular materials jam when the average density is higher than a critical value[2, 3, 4, 5]. Colloids aggegrate and form rigid structures when temperature is lower than the critical temperature[6, 7, 8]. All these phenomena are characterized by the fact that some sort of rigidity emerges when a physical parameter goes below its critical value, due to some kind of local interaction. Similar phenomena also arise in systems of weakly attractive particles[9, 10], foams[11, 12] and concentrated emulsions[13, 14]. Despite these physical systems are of very different nature, they exhibit a similar transition. We call this transition between a liquid state with zero rigidity (shear modulus for granular material) and a disordered jammed state with non-zero rigidity the jamming transition. In the light of universality, two natural questions arise: Are the underlying dynamics of these systems universal? Can we describe them by a minimal model capturing the essence of the dynamics?

While such a minimal model is still lacking in the field, the notion of a jamming phase diagram is introduced by Liu and Nagel[15]. The idea is to construct a phase diagram of two states - jammed and unjammed - using three physical parameters: density, temperature and load. A schematic diagram is shown in Figure 1. The jammed phase near the origin is bounded by the depicted surface. The choice of these particular axises are movitated

by the dynamics in different systems. For granular materials, the dynamics is essentially athermal, so temperature is not important. But because a large density can jam the system while a high shear stress can make the material flow, the density and load (shear stress in this case) are revelent to the jamming transition. Thus the two axises. The third axis, temperature, is easy to understand as thermal effect is important in jamming transition in colloid suspensions, for example.

Note that this phase diagram describes non-equilibrium phenomena because the load is tied to dynamics. The state of the system, specified by its configuration, is not enough to dictate weather the system jams. The load must be specified to give a complete description. This is different from equilibrium statistical mechanics.

The goal of this paper is to summerize the current development on the topic. We will discuss two systems, namely frictionless granular materials and colloid suspensions, to illusrate that the jamming phase diagram indeed unifies the description of these two systems.

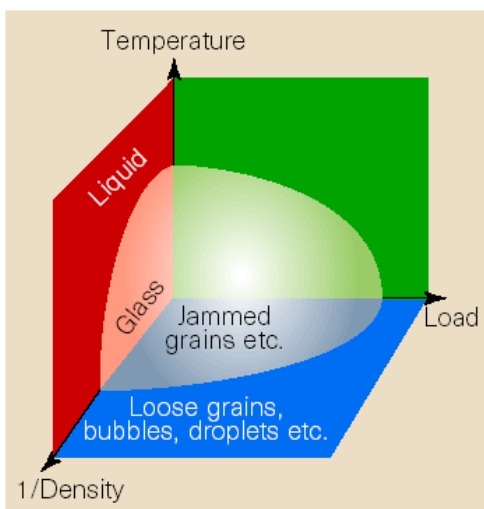


Figure 1: A schematic jamming phase diagram. The jammed phase near the origin is bounded by the depicted surface.

2 Weakly attractive colloidal particles

Trappe *et al.* [9] were one of the first reseachers to put the idea of a unified jamming phase diagram to an experimental test. They considered attractive colloidal particles, and treated the suspending fluid as an inert background. For this system, the density is set explicitly by the particle volume fraction, ϕ. The degree of thermalization is control by the parameter k_BT/U where U is the interparticle attractive energy. And the stress, σ is measured in units of $\sigma \equiv k_BT/a^3$ where a is the radius of colloidal particles.

To examine the generality of the concept, they used three colloid systems - carbon black, polymethylmethacrylate (PMMA), and polystyrene. For all these systems, a highly dispersed, fluid phase of particles is transformed into a jammed solid with increasing ϕ, increasing U, or decreasing σ.

For carbon black, a well-defined transition from fluid-like to solid-like structure is obtained by either increasing ϕ or U. The elastic modulus, G'_p, goes to zero when unjammed, while the viscosity, η diverges when the system jams. These results are shown in Figure 2.

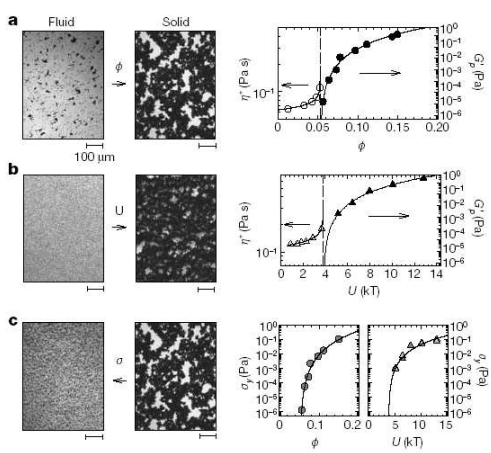


Figure 2: Control parameters for the jamming transition. On the left are the optical micrographs showing the jamming transition for carbon black by (a) increasing ϕ, (b) increasing U and (c) decreasing σ. As can be seen in (a) and (b), the viscosity, η, diverges as the system jams and the elastic modulus, G'_p goes to zero as the system unjams. (c) shows the phase boundaries bewteen jammed and unjammed phases for different values of stress σ, ϕ and U.

The data shown in Figure 2 can be fitted using power law functions

$$\eta = \eta_s(\phi_c - \phi)^{-\nu_\phi} \tag{1}$$

and

$$G'_p = G'_\phi(\phi - \phi_c)^{v_\phi} \tag{2}$$

where $\nu_\phi = 0.13$ and $v_\phi = 4.0$ are fitted. η_s, G'_ϕ and ϕ_c depend on U. Similar power law can be written in terms of U:

$$\eta = \eta_D(U_c - U)^{-\nu_U} \tag{3}$$

and

$$G'_p = G'_U(U - U_c)^{v_U} \tag{4}$$

where again $\nu_U = 0.13$ and $v_U = 4.0$ are fitted. η_D, G'_U and U_c depends on ϕ. The power law behavior reminds us about critical phenomena in equilibrium critical phenomena; however no rigorous connection is made because this is a non-equilibrium transition.

For any jammed configuration, if we increase the stress indefinely, at some point the system will be unjammed. This defines the yield stress, σ_y, which is the phase boundary. From the experimental data, Troppe *et al.* found a remarkable power law behavior of σ_y:

$$\sigma_y = \sigma_\phi(\phi - \phi_c)^{\mu_\phi} \tag{5}$$

and

$$\sigma_y = \sigma_U(U - U_c)^{\mu_U} \tag{6}$$

where $\mu_\phi = 3.4$ and $\mu_U = 2.4$ are fitted. These reults are shown in Figure 2. The phase diagram can be constructed from these results and is shown in Figure 3. Note that the curvature of the phase boundaries are different from the original proposal by Liu and Nagel[15]. Note also that there is no reason to expect the phase boundaries to follow a power law.

Similar scaling behaviors are found in PMMA and polystyrene, but the exponents are different. This feature is a clear sign that this transition is not a conventional equilibrium phase transitions. And special care has to be taken when we discuss the universality of the transition. We will see in the next section that the systems of frictionless particles bear the same feature, namely the exponents depends on the interaction potential, but not

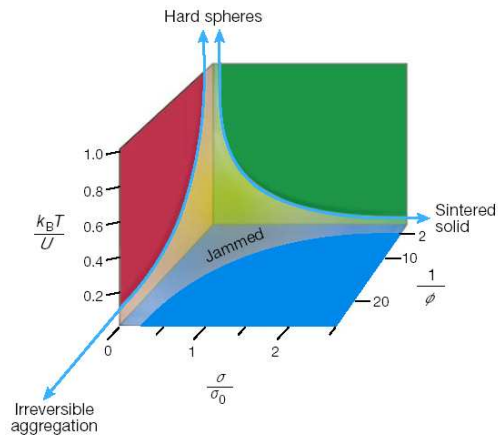


Figure 3: The jamming phase diagram for carbon black.

on dimensionality. How do we apply our knowledge of renormalization group and critical phenomena to these systems is still an open question.

The same relation between σ and ϕ is also found in mixture of oil and silica particles[16], supporting the idea of jamming phase diagram as a unified picture.

3 Systems of Frictionless Particles

As the first advocate of the jamming phase diagram, Nagel's group has studied jamming of frictionless particles at zero temperature and zero applied stress[3]. They used molecular dynamics simulation to simulate frictionless particles in a box, with interacting potential

$$V(r_{ij}) = \Theta(\sigma_{ij} - r_{ij})\epsilon(1 - r_{ij}/\sigma_{ij})^{\alpha}/\alpha, \tag{7}$$

where $\Theta(x)$ is the Heaviside step function, r_{ij} is the inter-particle distances, σ_{ij} is the sum of the particle radii, ϵ sets the energy scale and α is a parameter specifying the interaction. In particular, $\alpha = 3/2, 2, 5/2$ are chosen in their work. Simulations are done in both two and three dimensions, for both monodisperse and bidisperse systems. Because of the Heaviside step function, the particles only repel each other when they are overlapped. A sufficiently large value of ϵ was chosen to avoid large deformation of the particle sizes. All these settings are chosen to simulate systems of granular materials.

Technically, the definition of the jamming transition given earlier is not shape. Given a however small shear stress, all configurations will eventually flow, depending on the time scale of the experiments. So the jamming surface depicted in Figure 3 is typically not sharp. To make things precise, it is better to consider zero temperature and zero applied shear stress. The transition is then only controlled by the density of the system. Studies show that the transition in this case is sharp and well-defined in the limit of infinite system size[2, 3]. The critical point at zero temperature and zero shear stress is called the point J, which stands for jamming(figure 3).

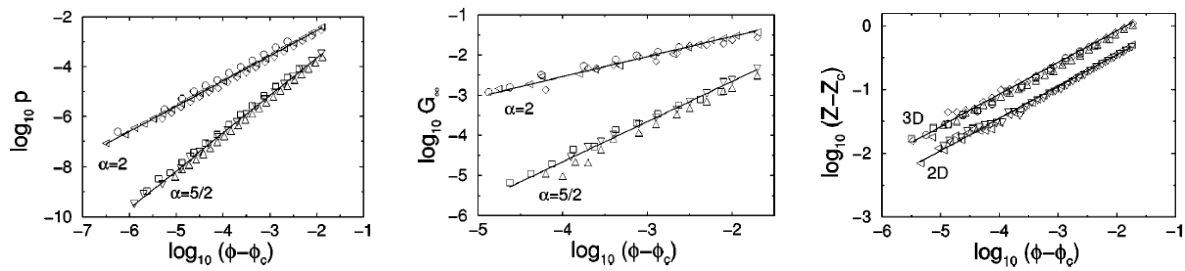
The physical meaning of the transition at point J is clear. At low density, the particles do not see each other and any small applied shear stress will move the particles. As the density is increased, the particles get closer and closer to each other. They are then 'maximally packed' at some critical density, meaning no particle can move independently with respect to the rest of the system. The system jams. And a nonzero yield stress emerged. This simple pictures shines light on the relation between jamming transition at zero temperature and the problem of random close packing[17, 18]. Indeed, the critical value $\phi_c = 0.639 \pm 0.001$[2, 3] is found to be very close to the random close packing density $\phi_{rcp} = 0.64$[17, 18]. This relation implies the jamming transition at zero temperature and zero applied shear stress is controlled by the geometry of packing.

3.1 Critical scaling

Power law behaviors are observed near point J for various physical quantities, including pressure p, static shear modulus G_∞, coordination number $Z - Z_c$, where Z_c is the coordination number at $\phi = \phi_c^+$ and the height, $g(r_0)$ and width, s, of the first peak of the radial distribution function $g(r)$. They are plotted in figures 4 and 5.

A remarkable point to note is the critical exponents for pressure and static shear modulus do not depend on dimensionality, but on α. This means that the exponents depend on the microsopic interaction potential between particles. And if we write

$$p = p_0(\phi - \phi_c)^\psi \quad \text{and} \quad G_\infty = G_\infty^0(\phi - \phi_c)^\gamma, \tag{8}$$

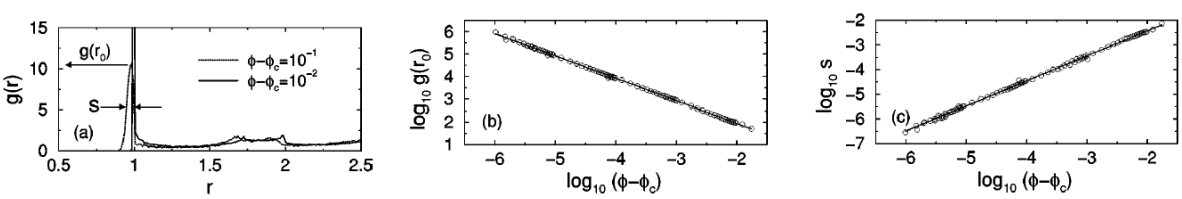


(a) Pressure. The upper(lower) line has slope of 1.0(1.5).

(b) Static shear modulus. The upper(lower) line has slope of 0.5(1.0).

(c) Coordination number. The slope is 0.50 for both lines.

Figure 4: Power law behaviors of various physical quantities below point J. For $\alpha = 2$ (harmonic repulsions), the symbols are 3D monodisperse (circles), 3D bidisperse (diamonds), 2D bidisperse (leftward triangles). For $\alpha = 5/2$ (Hertzian potential), they are 3D monodisperse (squares), 3D bidisperse (upward triangles) and 2D bidisperse (download triangles).



(a) The radial distribution function $g(r)$.

(b) The height, $g(r_0)$, of the first peak of $g(r)$. The slope of the solid line is -1.

(c) The width, s, of the first peak of $g(r)$. The slope of the solid line is 1.

Figure 5: Power law behaviors of the height and width of the first peak of the radial distribution function $g(r)$ for a $N = 1024$ monodisperse system with harmonic repulsions in 3D at $\phi - \phi - c = 0.1$ and 0.001.

it is not difficult to note the relations

$$\psi = \alpha - 1 \quad \text{and} \quad \gamma = \alpha - 3/2, \tag{9}$$

independent of dimensionality or polydispersity. This is different from what we expect and this posts questions about the universality of the underlying dynamics. The physical meaning of equation (9) is explained in ref. [3] and is not revelent to our current discussion.

The coordination number jumps from zero to some non-zero value Z_c discontinuously at point J. This can be understood as follows. For $\phi < \phi_c$, the system is dilute and there is no contact between the particles on average. At the onset of jamming, $\phi = \phi_c$, the system achieve a random close packing and every particle is in contact with some other particles. Thus a nonzero value of Z_c. This is related to the divergence of $g(r_0)$ as shown in figure 5(b). When $\phi < \phi_c$, the system is not jammed and $g(r)$ is a smooth function. But at $\phi = \phi_c$, the coordination number goes to Z_c discontinuously, this translates to a δ function at the radius of the particles in $g(r)$. And the area under the δ function should roughly be Z_c.

In order to have the area under the δ function equals to Z_c under scaling. The scaling of the height of the peak should counteract the scaling of the width of it, *i.e.,* $\eta = \Delta$, where η and Δ are defined by

$$g(r_0) = g_0(\phi - \phi_c)^{-\eta} \quad \text{and} \quad s = s_0(\phi - \phi_c)^{\Delta}. \tag{10}$$

This is verfied by the data ($\eta \sim \Delta \sim 1.0$).

3.2 A diverging length scale

Similar to critical phenomena, there is a diverging length scale associated with the jamming transition. If the system is held at a packing fraction slightly below the critical value, the system is unjammed and all particles can move and rearrange. However, the number of particles that must move in order to allow a rearrangement will depend on how close one is to the transition. In an infinite system, if one applies a fixed, infinitesimal velocity to a particle, it will move and disturb the rest of the system. The disturbance will extend to a distance ξ_T^-, the transverse length scale, in a direction perpendicular to the applied force. We expect that ξ_T^- will diverge as one gets close to the transition because as the density

approaches the critical value, the particles are closely packed. More and more particles must rearrange to allow for the single particle to move in the longitudinal direction. The idea is illusrated in Figure 6.

A careful study of this transverse length scale is carried out by Drocco *et al.* [5]. They solved a system of discs interacting through a harmonic potential in 2D. There is no external force acting on the particles except one. And they measured the probability distribution of velocity as the single particle is draged. They found that the number of moving discs is given by

$$n_{moving} = n_0(\phi - \phi_c)^{-\tau} \tag{11}$$

where $1.2 < \tau < 1.46$ is fitted. The data is shown in figure 7. Because the number of moving discs, n_{moving}, is proportional to $(\xi_T^-)^d$, we get $\xi_T^- = \xi_T^0(\phi - \phi_c)^{-\tau/2}$. This is the diverging length scale in the system.

Similar studies on the vibrational modes of the system is carried out by Silbert *et al.* [19].

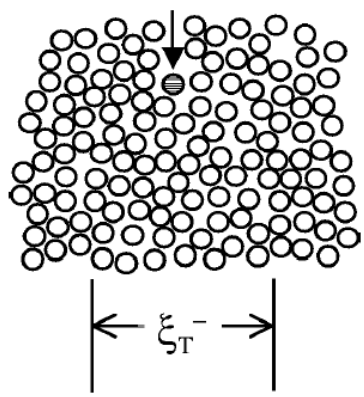


Figure 6: A sketch of the transverse length scale.

4 Discussion

The idea of a jamming phase diagram tries to unify different jamming systems. Up till now, colloidal systems and systems of frictionless particles both show consistency with the picture. Near the critical point, there are various power law dependences. There is also a diverging length scale, just like what we get for critical phenomena. But beside the suggestion of

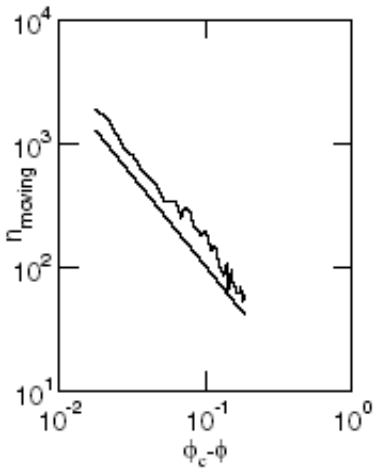


Figure 7: Plot of n_{moving} against $\phi_c - \phi$, showing a power law dependence.

universality between these systems and a qualitative prediction of the phase diagram, the jamming phase diagram does not provide any information about the physical system. In the frictionless particle systems, O'Hern *et al.* studied the point J, at which the jamming transition is well defined. But the critical exponent associated with that depends not on the dimensionality, but the microscopic interaction potential between particles. There is no idea how to understand this by conventional theories.

The motive behind the jamming phase diagram is universality. But it lacks a theoretical model to predict various behaviors. One way to unify these systems is to bulid a minimal model, capturing the local geometrical interaction between individual particles. The model may come from one particular system, but it should be general enough to be applied to other systems that jam. Because all of the jamming systems are disordered and break egrodicity, normal technique of statistical mechanics is not applicable. We can formulate the problem by using Edwards' postulate of statistical mechanics of powder[20, 21]. Its essence is to study a suitable ensemble averages over its 'mechanically stable' states. The resulting theoretical framework is the very similar to conventional statistical mechanics, with the thermal average replaced by the configurational average. If one write down a minimal model of free energy within this framework. Renormalization group can then be applied to the theory, and universal features of the systems can be extracted. Perhaps we will then understand what determines the exponents and why they have these particular dependence.

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