The Renormalization Group as a Method for Analyzing Differential Equations

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Abstract

By following hints derived from similarities between critical phenomena and the theory of qualitatively significant (i.e. singular) perturbations to ordinary differential equations, a renormalization group method of analyzing singular perturbations in differential equations is developed. This method is applied to the Raleigh Equation to develop the method explicitly. Other recent work is briefly discussed, and prospects for future work.

1 Introduction

Quantum theory, and equilibrium statistical mechanics have traditionally been the areas most influenced by renormalization group theory and phase transitions. But equilibrium statistical mechanics and quantum theory are only convenient descriptions for a fairly limited subset of interesting systems in nature. Equilibrium statistical mechanics, for example, has only narrow applications to dynamical aspects of systems, and in general fails when substantial dynamics are present. Many of the systems that are not conveniently described by quantum theory or equilibrium statistical mechanics find their most natural expressions as differential equations. Dynamical systems, environmental competition between species, predator-prey competition are all examples of diverse systems where differential equations are an extremely convenient description [9].

However, a careful study of differential equations reveals that there are many features and problems in their study that closely resemble the features and problems in statistical mechanics and field theory that renormalization group methods were designed to overcome. This paper will show how ideas from the renormalization group have been applied to differential equations by showing the similarity of divergences in perturbative expansions of certain types of differential equations to those in perturbative analysis of the Ising universality class. It will then be shown by carrying out a renormalization group calculation on a specific example that renormalization group methods can be used to render the perturbation theory for the differential equation finite and obtain a meaningful approximation to the solution valid over long time intervals[1, 4, 7].

In the Landau theory of continuous phase transitions, the partition function in the d dimensional Ising universality class can be written approximately as [2]:

$$Z = \int D\phi e^{-\int d^d r 1/2(\bigtriangledown\phi)^2 + 1/2 r_0 \phi^2 + u_0 \phi^4} \tag{1}$$

Under an appropriate rescaling, this can be rewritten with all of the temperature dependence of the coefficients moved to the fourth order term:

$$Z = \int D\varphi e^{-\int d^d x 1/2(\bigtriangledown\varphi)^2 + 1/2 r_0 \varphi^2 + \bar{u}_0 \varphi^4} \tag{2}$$

where all of the temperature dependence of the coefficients has been moved by re-scaling to the 4th order term, giving it a temperature dependence of

$\bar{u}_0 \sim t^{(d-4)/2}$ (see [2] for a careful analysis). This shows that a perturbative expansion of the functional integral in quartic powers for $d < 4$ is sure to diverge near $t = 0$ regardless of how small the constant u_0 is before rescaling. This is an indication of the fact that the inclusion of the higher order term introduces fundamentally new and qualitatively different physics near the critical temperature even when that term is very small. Physically, this means that the thermodynamics of the quadratic model cannot be smoothly deformed into the thermodynamics of the full model. For example, in the model above, the quadratic model alone only describes a trivial order parameter. When a quartic term is present, the quadratic term can be negative for some values of t and there can two non trivial values of the order parameter for $t < 0$. So perturbation theory would not be a one to one mapping, and naive perturbation expansions around the quadratic theory are clearly bound to fail because of multivaluedness, among other factors. In that sense, the u_0 term is (in the language of differential equations) a *singular perturbation*. However, use of renormalization group analysis removes the divergences from the perturbation expansion, and allows critical exponents to be calculated from this theory (see chapter 12 of [2]).

As suggested above, in the theory of ordinary differential equations, there exists an analogous situation. Some ordinary differential equations are formulated with perturbations that, like the quartic term in the Landau theory, introduce qualitatively new features in the solution. In some cases, solutions fail to even exist for the unperturbed problem [3]. Again, like in the Landau theory, the unperturbed solutions (if they exist) are not simply related to the solutions of the actual problem, as assumed by perturbation theory. Unfortunately, that simple relation between the unperturbed and full solution is required for naive perturbation expansions to converge. Thus, for similar reasons as in critical phenomena, these perturbations give rise to divergences in perturbation theory. Perturbations with these characteristics are called called *singular perturbations*. While a large array of mathematical techniques have been developed for obtaining approximate solutions valid for large ranges of times [3], most of these these techniques are of limited validity, or are difficult to carry out systematically for many problems, requiring ad hoc assumptions in many cases [1]. This points to a need for a systematic approach to analyzing singular perturbations. To find a method, it is reasonable to look to field theory and critical phenomena where similar problems have been solved, such as the perturbative expansion of the Landau theory discussed above. In those fields, the divergences of the perturbation expansion have been largely tamed by the use of renormalization group techniques. This suggests that such a scheme may work for developing a systematic approach to differential equations with singular perturbations. Such a method was developed by Goldenfeld and collaborators [1, 4, 2]. It will be developed below through the careful analysis of Raleigh's equation after some preliminaries are discussed. While the discussion will seem somewhat involved technically, it is both relatively simple and non-trivial, and thus provides a comparatively transparent way for exposing both the power and basic ideas of the renormalization group approach.

2 Perturbation theory for Differential Equations

Because standard perturbation theory for ordinary differential equations is not part of universal physics education, I will briefly review the technique below. It will be developed through an example and will follow Bender and Orsag [3], section 7.1 closely.

Consider the differential equation

$$y(x)'' = \epsilon f(x) y(x) \tag{3}$$

with boundary conditions $y(0) = 1$, $y'(0) = 1$. This equation can be solved perturbatively by expanding $y(x)$ as a series in ϵ, $y(x) = \sum_{n=0} \epsilon^n y_n(x)$, and solving recursively. For the zeroth order solution

$$y_0(x) = 1 + x \tag{4}$$

To get the 1st order result we plug the first order expansion into the differential equation and keep terms of order $O(\epsilon)$

$$\frac{d^2}{dx^2}(y_0(x) + \epsilon y_1(x)) = \epsilon(y_0(x) + \epsilon y_1(x)) \tag{5}$$

$$= \epsilon \frac{d^2}{dx^2} y_1(x) = \epsilon f(x) y_0(x) + O(\epsilon^2) \tag{6}$$

This leaves a differential equation for the first order result

$$y_1(x)'' = f(x) y_0(x) \tag{7}$$

Note that the boundary conditions for the first order result are determined by that fact that the zeroth order result satisfies the boundary conditions for the full problem. Thus the higher order terms have the boundary conditions $y_n(0) = 0$ and $y'_n(0) = 0$. This is a general property of such expansions. Eq. 7 is solved by straightforward integration with those boundary conditions to give the first two terms in the series for $y(x)$ as

$$y(x) = 1 + x + \int_0^x dx' \int_0^{x'} (1 + x'')f(x'')dx'' \tag{8}$$

The perturbation expansion for this equation is well behaved, and higher order terms are found easily [3]. This is an example of a regular perturbation, for which the perturbation series is well behaved (provided the function $f(x)$ isn't too pathological). Such straightforward perturbation methods fail for singular perturbations as mentioned above.

3 Introduction to Raleigh's Equation and Singular Perturbations

A prime example of a singular perturbation is Raleigh's equation. Raleigh's equation was introduced by Lord Raleigh to model the vibrations of the reed in clarinets [5] and is in the form of an oscillator with nonlinear damping

$$\frac{d^2y}{dt^2} + y = \epsilon \left(\frac{dy}{dt} - \frac{1}{3} \frac{dy}{dt}^3 \right) \tag{9}$$

It is well known that the long time solutions of Raleigh's equation approach a bounded limit cycle in phase space that looks like a distorted oscillator limit cycle, i.e., closed and roughly circular. However, because the perturbation introduces nonlinear terms, the straightforward method of expansion around the unperturbed solutions fails, as we will see below [3]. This will not stop us from using this expansion method, as Renormalization Group methods will be introduced in order to tame the divergences that will arise, and allow us to make sense of the perturbation expansion. This already gives a sense of a major advantage of RG methods for differential equations, that it can be used to make sense of easily obtained perturbation series that would otherwise be invalid [1, 4].

We now proceed with a naive perturbative expansion for Raleigh's Equation. Let $y(x) = \sum_{n=0} \epsilon^n y_n(x)$ and plug in to find an equation for the zeroth order term

$$\frac{d^2}{dx^2} y_0(x) + y_0(x) = 0 \tag{10}$$

We find the solution to Eq. 10 to be

$$y_0(x) = R_0 sin(t + \Theta_0) \tag{11}$$

We now plug this solution plus the unknown first oder term back into the differential equation as in the outline of perturbation theory above

$$\epsilon \frac{d^2}{dt^2} y_1(x) + \epsilon y_1(x) = \epsilon \left(\frac{d}{dt}(y_0(x) + \epsilon y_1(x)) - \frac{1}{3} \left(\frac{d}{dt}(y_0(x) + \epsilon y_1(x)) \right)^3 \right) \tag{12}$$

$$\rightarrow \frac{d^2}{dt^2} y_1(x) + y_1(x) = R_0 cos(t + \Theta_0) - \frac{1}{3}(R_0 cos(t + \Theta_0))^3 + O(\epsilon) \tag{13}$$

This is now a *linear* inhomogeneous equation for $y_1(x)$. This linearity of differential equations for the terms of the perturbation series is a general feature of this type of perturbation theory, as can be seen by noting that nonlinear terms are always of higher order in ϵ and are therefore dropped. This is generally a great improvement. Note, however, that Eq. 13 is equivalent to a driven oscillator equation for $y_1(x)$, and that the driving term has frequency components at the natural frequency of the oscillator equation. Thus the second order term $\epsilon y_1(x)$ will grow unbounded with time so that it is not small relative to the zeroth order term after some time has elapsed (such terms are called *secular* terms in the perturbation series), while $y_0(x)$ is periodic and bounded. This causes the perturbation series to diverge after time ha elapsed. Thus we can see without even solving the equation that the naive perturbation expansion will only work, if at all, for very short time scales.

4 Renormalization of Perturbation Theory for Raleigh's Equation

The problem of secular terms in the perturbation expansions around the unperturbed solution makes it appear that such an expansion is hopeless.

However, the analogous problem of divergence of perturbation expansion of the partition function around the free field theory in Eq. 1 has been solved using renormalization group methods [2] including the introduction of arbitrary length scales. This suggests that a similar analysis of the Raleigh equation may render the expansion around the free solution convergent and eliminate the divergent secular terms for all times. Analysis of this type has been carried out for the Raleigh equation and other examples by Chen, et. al., in [1, 4]. We will follow these papers very closely in the analysis that follows. The key idea mathematically is to mimic field theory by introducing an arbitrary parameter analogous to the ultraviolet cutoff in field theory, that can cancel the divergence. We then use renormalization group equations to ensure that the solution does not depend on the parameter, and use its arbitrariness to carry out the cancellation of the divergence.

The solution for the first order term in the perturbation expansion of the Raleigh equation for an arbitrary initial time t_0 is

$$y_1(x) = -\frac{R_0}{96} cos(t+\Theta_0) + \frac{R_0}{2}\left(1-\frac{R_0^4}{4}\right)(t-t_0) sin(t+\Theta_0) + \frac{R_0^3}{96} cos\left[3(t+\Theta_0)\right] + O(\epsilon^2) \tag{14}$$

Note the secular term. Following standard renormalization group methods (at least in statistical mechanics and field theory, where it corresponds to an ultraviolet cutoff rendering the theories finite), we introduce an arbitrary time scale τ and rewrite the secular term coefficient $t - t_0$ as $t - \tau + \tau - t_0$. We now introduce renormalization constants (in a process analogous to wave function renormalization) defined by

$$Z_1 = 1 + \sum_{n=1} a_n \epsilon^n,\ R(t_0) = Z_1 R(\tau) \tag{15}$$

$$Z_2 = \sum_{n=0} b_n \epsilon^n,\ \Theta(t_0) = Z_2(t_0, \tau) + \Theta(\tau) \tag{16}$$

Note this is a formal step. Nothing has happened except for a definition. We want to choose the constants a_1 and b_1 such that the terms with $\tau - t_o$ are eliminated to order ϵ. These choices can be viewed as simply constraints on the form of $R(\tau)$ and $\Theta(\tau)$. We examine the perturbation expansion and choose $a_1 = -\frac{1}{2}(1 - \frac{R^2}{4})(\tau - t_0)$ and $b_1 = 0$. These choices can be easily be checked by plugging them into the perturbation expansion and seeing that to order ϵ they yield the expression

$$y(t) = \left(R + \epsilon\frac{R}{2}\left(1 - \frac{R^2}{4}\right)(t-\tau)\right) sin(t + \Theta(\tau) - \frac{\epsilon R^3}{96}cos(t + \Theta(\tau)) + \frac{\epsilon R^3}{96}cos[3(t + \Theta(\tau))] + O(\epsilon^2) \quad (17)$$

However, just as in quantum field theory, the physics cannot depend on the ultraviolet cutoff, the dynamics of the Raleigh equation must not depend on the arbitrary parameter τ. So we have the renormalization group equation $\frac{\partial y}{\partial \tau} = 0$ for all t. Carrying out the τ derivative on Eq. 17 we find that this condition yields two renormalization group equations (an amplitude and a phase equation)

$$\frac{dR}{d\tau} = \frac{\epsilon R}{2}\left(1 - \frac{R^2}{4}\right) + O(\epsilon^2) \quad (18)$$

$$\frac{d\Theta}{d\tau} = O(\epsilon^2) \quad (19)$$

These expansions, can, of course, be carried out to higher order in ϵ, and can be solved as they are relatively easily (note they are first order). The solutions are

$$R(\tau) = \frac{R(0)}{e^{-\epsilon\tau} + 1/4R(0)^2(1 - e^{-\epsilon t})^{1/2}} + O(\epsilon^2 t) \quad (20)$$

$$\Theta(\tau) = \Theta(0) + O(\epsilon^2 t) \quad (21)$$

So if we use these functions in our solution, independence on τ is guaranteed to $O(\epsilon^2)$. But before we insert these equations back into the differential equations we have one more key step to make. We note that the even with our new, renormalized R and Θ we still have the secular term $t - \tau$ in the perturbative expansion. This is resolved by noting that we explicitly constructed the theory to be totally independent of the choice of τ, so for any range of time we are interested in, we can choose τ to be close to that time. Given that freedom, the simple thing to do is to set $\tau = t$ [1, 4, 6]. This eliminates the secular terms and renders the perturbation series convergent for long time behavior, where the naive expansion diverged. To specify a solution we need to choose initial conditions. To do so, we continue to follow [1] to the bitter end by choosing with Chen et. al. $y(0) = 0$ and $y'(0) = 2a$

yielding an approximation to the solution valid over all time scales

$$y(t) = R(t)sin(t) + \frac{\epsilon}{96}R(t)^3\left[cos(3t) - cos(t)\right] + O(\epsilon^2) \tag{22}$$

Note that the solution becomes periodic for long times, but with a distortion to circular motion in the phase plane caused by the $cos(3t)$ term. This is the expected behavior of the Raleigh equation. Higher order calculations and mathematical work indicate that no further inconsistencies arise [1, 7] .

Having completed this detailed analysis of Raleigh's equation, and having applied renormalization group ideas to render the perturbation series convergent, we are left with a number of technical issues to address. Perhaps the primary questions involve how generally such methods can be applied, how to estimate errors, and in what regimes are the solutions valid. These questions have been considered by several investigators. Kunihiro has formulated the theory in terms of the geometric concept of an envelope to a family of curves in an effort to provide a purely mathematical approach to this method, and has shown that the solutions are correct for a wide class of differential equations [6]. Additionally, M Ziane has given error estimates and shown that the solutions are valid within long time intervals, i.e. that the RG method is a global method [7]. In conclusion, the RG method developed above is robust over large time intervals, and is a systematic way to make sense of singular perturbation theory for ordinary differential equations [1, 4, 6, 7].

5 Conclusions and Prospects

While the calculations above seem quite technical, they are unified with the methods of renormalization in statistical physics, in spite of having no statistical component. Given the large similarities between the difficulties encountered in perturbation theory for field theory, critical phenomena, and singular perturbations terms in differential equations, it is not surprising that the powerful methods developed to analyze field theories and critical phenomena have been applied to differential equations.

The question then arises how deep the connections and analogies go. Recent work has shown that anomalous dimensions, key to the solution of the problem of critical exponents, arise in certain partial differential equations [2]. One can wonder if given that there are anomalous dimensions in differential equations (non-equilibrium descriptions) and that the equations can be handled using Renormalization Group methods, if perhaps other analogies exist. Perhaps a concept of phase could be developed to describe qualitatively different behaviors of solutions to differential equations. Another interesting possibility is the idea that perhaps renormalization group methods could be used to handle the divergences that arise in solutions of ill-posed integral equations. These divergences often rise from extraordinary sensitivity to initial data [8], which is a different origin than what gives rise to the secular terms in the differential equation case, but perhaps variances in initial data could be considered as singular perturbations and a theory could be developed to tame the divergences of ill posed integral equations using related Renormalization group methods.

In conclusion, work in recent years have shown that the renormalization group has far reaching applications. Recent progress in using RG methods for differential equations points to the possibility of being able to generate theoretical predictions over wide time scales for physical systems that have been very difficult to analyze previously. Given the rapid progress that has characterized the use of RG methods outside of traditional realms of statistical mechanics and field theory for the last several years, there are good prospects for further applications to be developed in the future.

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