Understanding Emergence of Cooperation using tools from Thermodynamics

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Abstract

Understanding how cooperation emerged in evolutionary biological systems is a long studied problem. To mathematically understand this phenomenon, simplified models of game theory are constructed and then analyzed usually with the help of simulations. Closed form solutions of when cooperation emerges even in such simplified models are hard to come by. This study will be primarily focusing on possible ways the problem of emergence of cooperation could be framed in thermodynamic language. Such a bridge will allow us to mine the extensive literature on critical dynamics in statistical mechanical systems, many of which have exact closed form solutions, for the purpose of understanding emergence of cooperation.

Contents

1 Introduction

According to Darwinian evolution theory of survival of the fittest, agents who act selfishly will be more successful in systems where others cooperate. Thus, the subsequent generations should only be populated by agents with selfish genes. However, we observe cooperation as a dominant strategy in several of the biological systems around us. For an individual agent, cooperation is a very expensive strategy unless everyone cooperates. Thus in such evolutionary biological systems one can expect a distinct "phase transition" from selfishness to cooperation.

Understanding this transition to cooperating behavior in complex biological systems is quite hard because of large number of variables involved. Instead, simplified mathematical models are studied where agents usually have fixed number of strategies to choose from (for example, cooperation and defection). Study of such mathematical models come under evolutionary game theory. By analyzing the conditions when cooperation emerges in such evolutionary games one can understand, upto first order, conditions in actual biological systems which might have resulted in plethora of cooperating strategies which we observe around us. However, solving even these evolutionary games analytically is often hard and we need to rely on agent based simulation methods to see how the system evolves.

Primarily in such evolutionary games we are interested in finding under what conditions cooperation dominates, or equivalently, we are interested in finding when a "phase transition" occurs between different strategies agents are using. Since phase transitions are an extensively studied topics in statistical physics, many recent studies have attempted to draw parallels (often successfully) between systems in game theory and systems in statistical mechanics. Such a connection will allow us to mine the extensive literature on critical dynamics in statistical mechanical systems, many of which have exact closed form solutions, for the purpose of understanding emergence of Cooperation. Thus, *the primary goal of this study is to understand the recent ways such parallels between statistical mechanical and game theory systems have be drawn.*

The layout of this study is as follows. In Section 2 we will briefly discuss the basic tenets of game theory and subsequently focus on a simple model of Prisoner's Dilemma between N players. In section 3, we look at three different strategies of finding equivalent thermodynamic systems. By applying these strategies on the N player Prisoner's Dilemma game introduced in Section 2, we will find latter two methods to be equivalent while the first gives incorrect results.Finally, we summarize in section 4 and conclude in section 5.

2 Evolutionary game theory

Game theory is the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. It is usually employed to understand problems in economics, psychology, political science, etc. All these systems have humans as the primary intelligent agents. However, counter-intuitively game theory tools could also be employed to understand evolution where agents are not rational[1]. The only important criteria is that the agents have a strategy which in evolutionary systems will be passed down genetically. Subsequently, genes corresponding to profitable strategy will be passed down. This study will be focusing only on systems where agents only have two strategies available to them: cooperation and defection.

Since, evolutionary games are just a subset of problems in game theory, we will start by first laying out a general setup for game theory problems and then look at a specific example of an evolutionary game. Problems in game theory are generally described with N players where each xth player having the ability to choose from $s_x = \{1, 2, 3, ...\}$ set of strategies available to him. Based on strategies adopted by everyone, each xth player receives a payoff determined by $u_x(\vec{s})$. Here $\vec{s} = \{s_1, s_2, ..., s_N\}$ denotes the strategy profile of all players. In usual game theory problems, each player will adopt a strategy which will maximize their payoff u_x in the next step. In special cases there exists a stationary state where no player will find it favorable to change their strategy. Mathematically such a state satisfies the following condition:

$$u_x\{s_1^*, s_2^*, ...s_x^*, ...s_N^*\} \leq u_x\{s_1^*, s_2^*, ...s_x', ...s_N^*\}, \qquad \forall x,\ \forall s_x' \neq s_x^*. \tag{1}$$

This is the so called *Nash Equilibrium.*

In evolutionary systems, payoffs can be interpreted as the fitness of the agent. The better its strategy, higher will be it's fitness (or payoff) and more chance it will have to propagate it's genes to next generation. However, in contrast to human systems, agents are not rational and the choice of strategy is guided by genetics. Mutations in genes or any other environmental factors could cause fluctuations in strategy adoption mechanism. In these cases rather than specifically choosing a strategy which maximizes their payoff, agents can be modeled to choose a strategy with a certain probability which in turn is some function f of the payoff:

$$p(s_x \to s_x') = f(u_x\{s_1, s_2, ...s_x', ...s_N\} - u_x\{s_1, s_2, ...s_x, ...s_N\}). \tag{2}$$

This "noise" around the optimum strategy can act as an equivalent *temperature* in such systems. In fact this equivalence becomes exact for Glauber dynamics (Ref.[2]) where the transition probability is given by

$$p(s_x \to s_x') = \frac{1}{1 + e^{\beta \Delta u_x}}, \tag{3}$$

$$\Delta u_x = u_x\{s_1, s_2, ...s_x', ...s_N\} - u_x\{s_1, s_2, ...s_x, ...s_N\}. \tag{4}$$

Here $1/\beta$ denotes the "noise" in the system. We will later see the importance of this dynamics in Section 3.3.

In such cases of probabilistic selection of strategies, a pure stationary state will not exist. However, for systems having large number of agents with identical payoff functions the final state would attain a stable value of parameters averaged over population. To find such final state agent based simulations are performed. A typical strategy of such simulations involves starting the system from an arbitrary (typically random) initial state. At every iteration step an agent x is chosen randomly and is allowed to change his strategy with some probability, Eq.2. Once the system becomes stable, parameters of interest can be calculated by averaging over the population.

2.1 Evolutionary game with N players and two strategies

Before describing the case for N players let us first look at a simple case with two players. In fact, A staple problem in game theory consists of a simple model with only two participating players with each having two choices: cooperation(C) or defection(D). The payoff in such a simple situation can be represented in the matrix form:

$$\begin{matrix} & \begin{bmatrix} C & D \end{bmatrix} \\ \begin{bmatrix} C \\ D \end{bmatrix} & \begin{bmatrix} R & S \\ T & P \end{bmatrix} \end{matrix} \tag{5}$$

where row (C, D) denotes strategy choices of the player whose payoff we are determining and column (C, D) denotes the strategy choices of the other player. R denotes the reward obtained by the player when both them cooperate. S is the sucker's cost payed by the player when it cooperates while the other defects. S is usually negative. T is the temptation profit obtained if the player defects and the other cooperates. And P is the punishment price payed if both the players defect. The Payoff matrices is same for both players and such games are known as symmetric games.

For evolutionary systems, we would primarily be interested in cases having a dilemma. For instance, the Prisoner's Dilemma game where the payoff parameters satisfy the condition $T > R > P > S$, is quite popular. This is because the Nash equilibrium for this game is given by a state where both players defect even though the combined utility of both players is maximized when they cooperate. Such kinds of dilemma are rampant in evolutionary systems where selfish acts provides a better payoff than cooperation.

Since evolutionary systems have large number of agents, we shall work with a simple variant of above game extended to N agents. Each agent is placed on a one dimensional ring[1]. The payoff for these agents is determined by the strategies they and the neighbor on the right chooses and the value is given by the same payoff matrix Eq.5. In the case of Prisoner's Dilemma, the Nash equilibrium state is when all the agents are defecting. However, if we further add noise into the system by having a probabilistic strategy selection, then the final state would then contain some players cooperating. In such systems, we would be primarily

[1] They are placed on ring to get a periodic boudary condition.

interested in finding how many agents are cooperating relative to defecting players. This is parametrized by

$$m = \frac{P_C - P_D}{N}, \tag{6}$$

where P_C and P_D denote the number of agents cooperating and defecting respectively.

Thus in such evolutionary games, we want to find the conditions under which m flips sign. These conditions will mark the boundary of a phase transition.

3 Thermodynamic analogues of evolutionary game theory problems

While translating evolutionary games to physics, the first step involves converting the agents to particles and subsequently interpret the strategies available to each agent as different possible states of particle. In the case of two available strategies: cooperation and defection, we shall think of the corresponding particle having spin up, $\sigma = +1$, (for cooperation) or spin down, $\sigma = -1$, (for defection). Hence, the order parameter m as given in Eq.6 becomes the average magnetization per particle in our system. Furthermore, the noise in selection strategy could then be interpreted as "temperature" in the thermodynamic system.

With the above parallels in place, the main goal would then be to find an equivalent Hamiltonian for our system of particles such that the equilibrium quantities (like magnetization) calculated using thermodynamic tools should be close to the values obtained from simulations employing Glauber Dynamics.

A quick check of our thermodynamic calculations can further be performed by taking the zero temperature limit ($\beta \to \infty$). In this limit, the thermodynamic state should be the Nash equilibrium state of our evolutionary game.

In the following sections we will look three seemingly different procedures of finding the equivalent Hamiltonian for an evolutionary game. As a demonstration we will apply the procedures on the game introduced in Section 2.1. Firstly, we shall look at the method employed by Adami and Hintze Ref.[3], where they model Hamiltonian using the payoff matrix of only a single agent. We shall then show the inconsistencies in this method. Following that we will look at the method employed by Sarkar and Benjamin Ref.[4], where they detail a process of finding an equivalent "Ising game" for any payoff matrix of form Eq.5. The obtained "Ising game" is then be solved using thermodynamic tools. Finally, we look at a much more intuitive method of finding the equivalent Hamiltonian in the third section. In fact, the method used by Sarkar and Benjamin is a more convoluted version of the third scheme. We shall see that this third scheme has in fact been shown to correctly predict the simulation results.

3.1 Method by Hintze and Adami

Let's start by first considering the Prisoner's Dilemma game between only two players. Adami and Hintze define interacting energy matrix for each player by flipping the sign of their individual payoff matrices.

$$E = \begin{bmatrix} -R & -S \\ -T & -P \end{bmatrix}. \tag{7}$$

The idea is that in thermodynamical systems the equilibrium state will try to minimize its energies. Thus the equilibrium state achieved using thermodynamical tools should coincide with the Nash equilibrium state which maximizes payoff.

For a N such players ($2 \times 2.... \times 2$), one way to construct the Hamiltonian of the whole system is by adding up energy matrix of all individual particles[2]. However, this would give inaccurate results. Note that the thermodynamic system will minimize the energy of our system. If we feed in a Hamiltonian which is a summation of payoff of all the individual players, then the equilibrium obtained from the thermodynamic method will have a state where total payoff is maximized (Pareto-Optima) and not the Nash equilibrium state where individual payoff is maximized.

Hence to maximize the individual payoff, Adami and Hintze only consider the energy matrix of a single player Eq.7, even for a N player system. The corresponding Partition function will be given by

$$Z = \sum_{states} e^{-\beta H_{state}} = e^{\beta R} + e^{\beta T} + e^{\beta S} + e^{\beta P}. \tag{8}$$

Since all the lattice points are equivalent, the net average magnetization over the whole lattice is same as the thermodynamic average of a single lattice point given by:

$$m = \langle J_z \rangle = \frac{1}{Z} \sum_{states} J_{z,state} e^{-\beta H_{state}} = \frac{1}{Z}(e^{\beta R} + e^{\beta S} - e^{\beta T} - e^{\beta P}) = \frac{e^{\beta R} + e^{\beta S} - e^{\beta T} - e^{\beta P}}{e^{\beta R} + e^{\beta T} + e^{\beta S} + e^{\beta P}}. \tag{9}$$

In the limiting case of $\beta \to \infty$ (zero "noise"),

$$m\Big|_{\beta\to\infty} \approx \frac{-e^{\beta T}}{e^{\beta T}} = -1. \tag{10}$$

This corresponds to the case of all states with spin down. This magnetization coincides with the Nash equilibrium state magnetization in Prisoner's Dilemma where all players are defecting.

Adami and Hintze [3] extended this same formalism to a more generic problem of public goods game where the payoff matrix of every player is determined by both the neighbors on

[2] Exact procedure to do so can be found in the paper Ref.[3].

left and right side in a 1D lattice. They found their thermodynamic parameter m to closely resemble the values obtained from their agent base simulation method based on Glauber dynamics.

3.1.1 Critique

A major critique for this method would be that it does not considers the choice of neighbor. In fact, the correct magnetization result for $\beta \to \infty$ was a coincidence and the method in reality does not work. We can see this by simply finding the average payoff received by every player in the zero noise limit. According to Adami and Hintze's method, this average payoff should be equivalent to finding the thermodynamic total energy of our Hamiltonian. If we consider the case of Prisoner's dilemma game discussed above, the average thermodynamic energy will be given by

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{Re^{\beta R} + Te^{\beta T} + Se^{\beta S} + Pe^{\beta P}}{e^{\beta R} + e^{\beta T} + e^{\beta S} + e^{\beta P}} \rightarrow_{\beta=\infty} T. \tag{11}$$

However, a player can only get T payoff when his neighbor cooperates which we know is not possible in the Nash equilibrium when everyone is defecting!

Another very serious critique of the paper is that they incorrectly employed their own method when solving for Prisoner's Dilemma game. Firstly, they do not flip the sign of payoff matrix to find the energy matrix as we did in Eq.7. Furthermore, they proceeded to find the Nash equilibrium state with a Hamiltonian constructed by summing the payoffs of all the N players. As we discussed earlier, such a Hamiltonian will lead to thermodynamic state corresponding to Pareto Optimum rather than Nash equilibrium. However, Adami and Hintze they do not seem to repeat the same mistake when calculating thermodynamic state for public goods game.

Final verdict is that this is a faulty method and it seems surprising that they were able to get this method to closely match the simulation results for Public Goods game.

3.2 Method by Sarkar and Benjamin

In their paper [4], Sarkar and Benjamin noted that the Adami and Hintze'e thermodynamic equivalent state for N player Prisoner's Dilemma gave incorrect answer when analyzed for limiting cases of different payoff matrices [3]. Correspondingly, they found a thermodynamic equivalent state for the same game by extending the analogy between Ising game and 2 player 2 strategy game as shown in [5] to N players. In what follows we will briefly go through their methodology.

[3] However, they do not mention why Adami and Hinze's method fails as we did in previous section.

The basic premise behind their method is that the Nash equilibrium for 2×2 game remains unchanged under the transformation of payoff matrix of the form:

$$U = \begin{bmatrix} R & S \\ T & P \end{bmatrix} \rightarrow \begin{bmatrix} R-\lambda & S-\mu \\ T-\lambda & P-\mu. \end{bmatrix} = U' \tag{12}$$

For $\lambda = \frac{R+T}{2}$ and $\mu = \frac{S+P}{2}$, we get

$$U' = \begin{bmatrix} \frac{R-T}{2} & \frac{S-P}{2} \\ \frac{T-R}{2} & \frac{P-S}{2} \end{bmatrix}. \tag{13}$$

Now consider a 1D Ising Hamiltonian of a two spin system,

$$\mathcal{H} = -2J\sigma_1\sigma_2 - h\sigma_1 - h\sigma_2 \tag{14}$$

where $\sigma_i = \pm 1$ denote the spins. For this Hamiltonian, we can write the energy contributed by spin σ_1 as

$$E_1 = \begin{bmatrix} -J-h & J-h \\ J+h & -J+h \end{bmatrix}. \tag{15}$$

Here the columns represent the spin state of $\sigma_2 = (+1, -1)$ and rows represent the spin state of $\sigma_1 = (+1, -1)$. We can now formulate a game where each player gets a payoff given by $-E_i$. Such games are known as Ising games. Moreover such games have an additional interesting feature that the Nash equilibrium state and the Pareto Optimum state coincide (both players want to cooperate)! So the configuration minimizing the Hamiltonian given in Eq.14 will also correspond to the the Nash equilibrium state.

We can see that the single player transformed payoff matrix obtained in Eq.13 resembles the payoff of a Ising game. Hence, the Nash equilibrium for any symmetric 2×2 game is the same as that of Ising game for

$$J = \frac{R-T+P-S}{4} \qquad h = \frac{R-T+S-P}{4}. \tag{16}$$

Thus by using thermodynamic tools on the Ising game we can find the equilibrium Ising state which is equivalent to the Nash equilibrium state of our primary game:

$$\text{Nash equilibrium}_{R,S,T,P} = \text{Nash equilibrium}_{J,h} = \text{Pareto optimum}_{J,h} = \text{Thermodynamic equilibrium}_{J,h}. \tag{17}$$

Sarkar and Benjamin extended this equivalence to a system of N agents playing prisoner's dilemma game as described in Section 2.1. The equivalent Ising Hamiltonian for this system will then be given by

$$\mathcal{H} = -\sum_{i=1}^{N} J\sigma_i\sigma_{i+1} - \sum_{i=1}^{N} \frac{h}{2}(\sigma_i + \sigma_{i+1}). \tag{18}$$

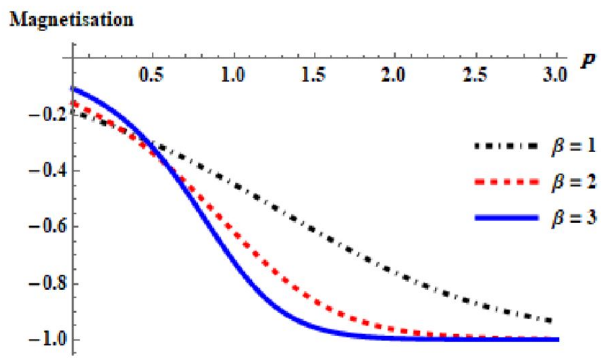

Figure 1: Figure take from Ref.[4]. Variation of Magnetization (m) with the punishment P for Prisoners dilemma for $T = 5$, $S = 0$ and $R = 3$. Lowering punishment from 1 to .5 increases the number of cooperators by 30%.

The partition function for this theory is given by

$$Z = e^{N\beta J}\Big(\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\Big)^N. \tag{19}$$

Correspondingly we can then use this partition function to calculate the average magnetization per lattice site in the system,

$$m = \frac{1}{N}\langle J_z\rangle = \frac{1}{N}\frac{\partial \log Z}{\partial \beta h} = \frac{\sinh \beta h}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}} \tag{20}$$

In the case of Prisoner's Dilemma ($T > R > P > S$), using relation from Eq.16, we have $h < 0$ and $J > 0$. Thus in the limit of $\beta \to \infty$:

$$m \approx -1. \tag{21}$$

Hence, we get back the Nash equilibrium state of all players defecting (spin down) in the zero noise limit.

To check that this is not a false state as the one we observed for previous method in Section 3.1.1, we further calculate the average thermodynamic energy per particle,

$$E = \frac{1}{N}\langle \mathcal{H}\rangle = -\frac{1}{N}\frac{\partial \log Z}{\partial \beta} = -J - h\frac{\sinh(\beta h) + \frac{1}{2}\frac{\sinh(2\beta h) - 4e^{-4\beta J}J/h}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}. \tag{22}$$

In the limit $\beta \to \infty$ for $h < 0$ and $J > 0$,

$$E \approx -J + h. \tag{23}$$

From Eq.15, we can see that the above energy is obtained when both neighboring particles have spin down. Thus, their result do not contradict the Nash equilibrium state.

Fig.1 shows the variation of m (Eq.20) with punishment P. At low temperatures (high β) the population consists mostly of defectors (negative magnetization) as this is the Nash equilibrium. Since, 1D Ising model does not have a phase transition at finite temperature we do not see emergence of cooperation in this system. This is a well-known result for Prisoner's dilemma in 1D.

Unlike Adami and Hintze's method, this method gives the correct result of Nash equilibrium in the limit $\beta \to \infty$. However, they do not perform any simulations to compare their results at finite Temperature.

3.3 Generic method for Potential Games

The method shown in the previous paper in fact falls under a generic scheme employed for special class of games known as *potential games*.

Potential games are systems on which a function can be defined:

$$V : \vec{s} \to \mathbb{R}$$

satisfying the relation

$$V(s_1, s_2, ...s'_x, ...s_N) - V(s_1, s_2, ...s_x, ...s_N) = u_x\{s_1, s_2, ...s'_x, ...s_N\} - u_x\{s_1, s_2, ...s_x, ...s_N\} \quad \forall x. \tag{24}$$

For such systems Nash equilibrium can simply be found by maximizing this potential V. Thus, we have a ready-made connection available to thermodynamic system with Hamiltonian simply defined as

$$\mathcal{H} = -\sum_{\{\vec{s}\}} V(\vec{s})|\vec{s}\rangle\langle\vec{s}|. \tag{25}$$

Moreover it has already been shown that equilibrium state achieved in simulations employing Glauber dynamics is exactly same as the thermodynamic equilibrium state of Hamiltonian described in Eq.25 Ref.[2].

The case of symmetric 2×2 player games studied in the previous two sections is actually a a form of potential game. There is a general framework of finding the potential which has been described in Ref.[6]. Here we will simply state the resulting potential for the payoff matrix Eq.5

$$V(s_1, s_2) = \frac{R - T + P - S}{2} s_1 s_2 + \frac{R - T + S - P}{4}(s_1 + s_2). \tag{26}$$

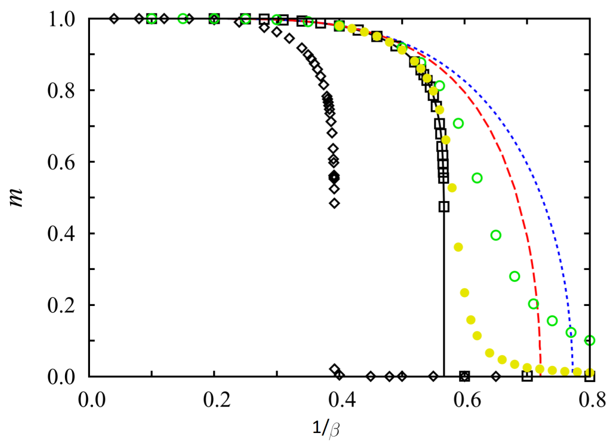

Figure 2: Figure take from Ref.[6]. Monte Carlo results for the order parameter m (magnetization) as a function of noise ($1/\beta$) for evolutionary potential games with Glauber dynamics on square lattice at several values of payoff parameters: in the notation of social dilemmas (with (R, P) fixed at $(1, 0)$): $T = 1.5, S = 0.5$ (boxes); $T = 1.4, S = 0.3$ (diamonds); $T = 0.5, S = 0.49$ (open circles); and $T = 0.5, S = 0.499$ (closed circles). The exact solution for the corresponding Ising model Eq.27 is denoted by a solid line. Dotted (blue) and dashed (red) lines illustrate the prediction of the cluster variation method for the levels of one- and two-site approximations.

The similarity of this potential with the Ising Hamiltonian Eq.14 is clearly apparent with J and H given by the same relation Eq.16 obtained in the previous section. Hence, this scheme coincides with the method employed by Sarkar and Benjamin.

Building upon this analogy, Ref.[6] analyzed a system of players situated on a 2D square lattice interacting with their nearest neighbors with the same payoff matrix Eq.5. They simulated this game using the Glauber dynamics for several values of (T, S) fixing $(R, P) = (1, 0)$. Moreover, the equivalent Ising game for the case $(S, T) = (0.5, 1.5)$ has an exact solution $(h = 0)$ given by Onsager:

$$m = \left(1 - \left[\sinh\left(\log(1+\sqrt{2})\frac{\beta}{\beta_C}\right)\right]^{-4}\right)^{\frac{1}{8}} \tag{27}$$

The exact solution for above (S, T) values accurately predicted the results from simulation as can be seen in Fig.2.

Note that the value $(S, T) = (0.5, 1.5)$ considered above does not correspond to a Prisoner's Dilemma game. In fact the Nash equilibrium for these value corresponds to everyone cooperating and hence is not an interesting system from the perspective of evolutionary biology. The only reason we considered this case was because the equivalent Hamiltonian was exactly solvable. Still, the results obtained are interesting. We can see that there is a critical temperature after which there is a phase transition to a system with equal number of cooperators and defectors.

4 Summary

In this study we went over three methods of finding an equivalent Hamiltonian for a given evolutionary game. We saw them in action for Prisoner's Dilemma game played among N agents situated on a 1D ring. Adami and Hintze's method, as detailed in Ref.[3], calculated the Hamiltonian by considering the payoff of only a single player. This naturally lead to incorrect results as the whole premise of problems in game theory requires taking into account the behavior of other agents. Following that we looked at Sarkar and Benamin's method, as detailed in Ref[4], where they made use of invariance property of Nash equilibrium to convert Prisoner's Dilemma game into an "Ising game" which then solved using thermodynamic tools. Finally, we discussed a much more intuitive (and popular) way of formulating Hamiltonian for evolutionary potential games, details of which have been covered in Ref.[6]. This method gave the same result as that of Benjamin and Sarkar.

We also witnessed the power of thermodynamic tools as we borrowed the exact solutions from the Ising models to accurately predict the behavior of the evolutionary games simulated using Glauber dynamics. We saw that absence of cooperation in 1D Prisoner's Dilemma game can be understood as the corresponding absence of phase transitions in 1D Ising models. For a special two strategy game in 2D, we saw that there existed a critical temperature (noise) for phase transition which could be predicted by Onsager's exact solution of the Ising model in 2D.

5 Discussion

In this study we focused on applying thermodynamic tools on a system where agents are fixed on a lattice site and cannot change their neighbors. In reality, usually agents can move and 'interact' with different agents over the course. Allowing this freedom of motions brings in an additional parameter of spatial structuring of the system. In fact it has been shown that under certain circumstances, motion could lead to cooperation in evolutionary Prisoner's Dilemma Ref.[7, 8]. For some cases this motion can also be taken into account by enlisting the aid of thermodynamics, specifically the kinetic theory of gases as done in Ref.[9].

However it should be noted that application of this thermodynamic method is restricted to narrow set of evolutionary games on which a potential exists. Moreover, thermodynamic results will fail if we deviate from Glauber dynamics. For instance, there are alternate strategy selection mechanism like imitation dynamics[4] which are of significance in biological systems but have no current equivalent interpretation in statistical mechanics. Other such variations of evolutionary games pose similar problem for out thermodynamic approach.

[4]In this case the agent chooses the strategy of its most successful neighbor with some probability.

Even with the above mentioned constraints on thermodynamic applications there are plenty of problems where thermodynamic approach could provide better results. One such problem which we think has not been studied so far is the case when an agent has a continuum of strategy between cooperation and defection to choose from. The corresponding statistical mechanic system would be described by a particle with a continuous spin and should be solvable.

Conclusion: While thermodynamic methods may never be able to accommodate the vast number of conditions applicable in evolutionary systems, they should be able to new insight on problems which are translatable to thermodynamic systems. Moreover application of statistical methods on such unconventional systems could also lead to development of new statistical tools.

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