Odd Viscosity in Chiral Active Matter: Theory and Experiments

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Chiral active matter is a hydrodynamic phase composed of self-spinning microscopic constituent particles. The non-equilibrium steady states of chiral active matter realize a novel dissipationless transport coefficient called the odd viscosity. In this paper I provide an overview of theories regarding the formation of odd viscosity in chiral active matter, both from a top-down hydrodynamic perspective and from a bottom-up microscopic perspective. Fluids with an odd viscosity support exotic topological waves and surface flows not found in conventional fluids. I briefly discuss the theory underlying these novel features and present experiments that observe these effects in two very different chiral active fluids.

INTRODUCTION

Chiral active matter is composed of particles that continuously inject both energy and angular momentum into the system. These exotic materials have a myriad of physical realizations, ranging from molecular motors at the smallest scale [1, 2] to macroscopic grains driven by some external force [3]. It is not hard to imagine that active chiral matter could have important technological applications, so it is important to understand the excitations and responses of their exotic non-equilibrium steady states. These properties are encoded in the constitutive relations of the material: equations expressing the stress tensor as a function of the strain and strain rate. The defining characteristic of chiral active matter is the presence of a time-reversal and parity symmetry-breaking dissipationless response coefficient $\eta^0_{ijkl} = -\eta^o_{klij}$ called the odd viscosity, which relates the stress σ_{ij} to the strain rate $u_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$. In Section 2 I present theory explaining how this odd viscosity arises from studying the linear response of chiral active matter around a non-equilibrium steady state. In Section 3 I discuss a theory taking the opposite approach, deriving a hydrodynamic response with odd viscosity from a microscopic model. In Section 4 I overview some of the consequences of odd viscosity, and in Section 5 I exhibit two experiments realizing some these consequences.

HYDRODYNAMIC DERIVATION

Consider a fluid described by the fields ρ, s, ℓ, and v_i, describing the mass density, entropy density, angular momentum density, and velocity. In this section I present a novel variational derivation of the equations of motion for these fields that obtains an odd viscosity term [4]. The starting point is the following action:

$$S = -\int d^x dt \left[\xi_0 + v_i\xi_i - \frac{\rho v_i v_i}{2} + \epsilon(\rho, s, \ell) - \omega\ell\right]. \tag{1}$$

The free energy ϵ obeys the standard thermodynamic relation $p = \mu\rho/m + sT + \Omega\ell - \epsilon$, where p is the pressure and Ω is an intrinsic angular momentum. The field ω is given by half the usual definition of the fluid vorticity, $\omega = \frac{1}{2}\epsilon_{ij}\partial_i v_j$. The fields ξ_μ are built from auxiliary fields called Clebsch parameters [5],

$$\xi_\mu = \rho\partial_\mu\theta + s\partial_\mu\eta + \ell\partial_\mu\phi + \Theta_\alpha\partial_\mu\Psi_\alpha. \tag{2}$$

These equations are standard with two exception: the presence of final term in the Lagrangian that couples the vorticity to the intrinsic angular momentum and the unusual Clebsch parameters Θ and Φ. Varying the action with respect to the fields θ, η, and ϕ produces the standard continuity equations for mass, entropy, and angular momentum density:

$$\partial_t \rho + \partial_i(\rho v_i) = 0, \quad \partial_t s + \partial_i(s v_i) = 0, \quad \partial_t \ell + \partial_i(\ell v_i) = 0. \tag{3}$$

By further varying the action with respect to the remaining fields, one additional equation can be obtained:

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) = \partial_j \left[-p\delta_{ij} + \sigma_{ij}^{\mathrm{odd}} \right]. \tag{4}$$

The stress tensor in this equation contains an odd viscosity,

$$\sigma_{ij}^{\mathrm{odd}} = \eta^0 \left(\partial_i \epsilon_{jk} v_k + \epsilon_{ik} \partial_k v_j \right), \quad \eta_0 = \frac{1}{2}\ell, \tag{5}$$

which appears because of the inclusion of the coupling between vorticity and intrinsic angular momentum. The three conservation laws for the mass, entropy, and angular momentum combined with the equations of motion for the momentum density constitute a complete set of five equations for the five hydrodynamic fields.

The above equations of motion are dissipationless because the action from which they were derived is time-translation invariant. A more general hydrodynamic description includes gradient corrections to currents (i.e. $\rho v_i \to \rho v_i + J_i^{\rho}$), energy and torque injection, and damping. The form of these terms is restricted by enforcing that the entropy production is always positive and energy is conserved (besides work done by external forces), but the resulting theory is still very complex. To make the equations of motion more tractable, the authors of ref. [4] neglect thermal effects, temperature dependences, gradient corrections, and any dynamics in the entropy density. The result is the following set of hydrodynamic equations that generalize those derived above:

$$\begin{gathered}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\
\rho(\partial_t v_i + v_j \partial_j v_i) = \partial_j \sigma_{ij} - \Gamma^{\nu} v_i \\
I(\partial_t \Omega + v_i \partial_i \Omega) = \tau + D^{\Omega} \nabla^2 \Omega - \Gamma^{\Omega} - \epsilon_{ij} \sigma_{ij} \\
\sigma_{ij} = \epsilon_{ij} \frac{\Gamma}{2}(\Omega - \omega) - p\delta_{ij} + \eta_{ijkl} v_{kl} + \frac{\ell}{2} \left(\partial_i \epsilon_{jk} v_k + \epsilon_{ik} \partial_k v_j \right).
\end{gathered} \tag{6}$$

The first is the continuity equation for mass and the second is the equation of motion for the momentum density, where the coefficient Γ^{ν} is a frictional damping term. The third is the equation of motion for the intrinsic angular momentum density $\ell = I\Omega$, where D^{Ω} and Γ^{Ω} are diffusion and damping coefficients respectively. The right-most term in the third equation accounts for the coupling between the intrinsic and angular orbital momenta. The final equation is the definition of the stress tensor. The first term represents the relaxation of the vorticity caused by friction between the microscopic rotating particles, the second term is simply the pressure, the third is the normal even viscosity, and the final term is the dissipationless odd viscosity discussed above. The presence of the odd viscosity is the key result of the derivation.

MICROSCOPIC HAMILTONIAN DERIVATION

The previous section obtained the odd viscosity as a transport coefficient by a variational calculation including an ad-hoc coupling between the vorticity and the intrinsic angular momentum. It was shown in a recent paper that odd viscosity can also be derived from the Hamiltonian mechanics of a very simple microscopic model of a molecular fluid [6]. Here I reproduce the outline of the calculation, showing that the odd viscosity not only *can* appear in the description of fluids, but *should* be present in a broad class of fluids. Consider a fluid of molecules with center of mass (CM) positions $\mathbf{r}^{\alpha}$ where each molecule consists of multiple point masses $m^{\alpha\mu}$ at positions $\mathbf{r}^{\alpha\mu}$ with momenta $\mathbf{p}^{\alpha\mu} = m^{\alpha\mu}\dot{\mathbf{r}}^{\alpha\mu}$. Here α and μ are particle and point mass labels. The total momentum density of the fluid is

$$g_i(\mathbf{r}) = \sum_{\alpha\mu} p_i^{\alpha\mu}\delta(\mathbf{r}-\mathbf{r}^{\alpha\mu}) \approx \sum_{\alpha} p_i^{\alpha}\delta(\mathbf{r}-\mathbf{r}^{\alpha}) - \sum_{\alpha\mu} p_i^{\alpha\mu}(\mathbf{r}^{\alpha}-\mathbf{r}^{\alpha\mu})_j\nabla_j\delta(\mathbf{r}-\mathbf{r}^{\alpha}), \tag{7}$$

where the approximation is a Taylor expansion of the Dirac delta function about the particle CM position. The first term on the right-hand side is the CM momentum density $\mathbf{g}^c$ and the second term is the "spin" momentum density $\mathbf{g}^s$. Through some manipulations, the spin momentum density can be expressed as

$$\begin{aligned} \mathbf{g}^s(\mathbf{r}) &= \frac{1}{2}\nabla\times\boldsymbol{\ell} + \nabla\cdot\mathbf{A} \\ \boldsymbol{\ell} &= \epsilon_{ijk}\sum_{\alpha\mu}(\dot{\mathbf{r}}^{\alpha\mu}-\dot{\mathbf{r}}^{\alpha})_j(\mathbf{r}^{\alpha\mu}-\mathbf{r}^{\alpha})_k\delta(\mathbf{r}-\mathbf{r}^{\alpha}), \end{aligned} \tag{8}$$

where $\boldsymbol{\ell}$ is the angular momentum density and $\mathbf{A}$ is a complex expression playing the role of an alignment tensor order parameter. This derivation assumes isotropy (beside the presence of angular momentum), which the order parameter disobeys, so $\mathbf{A}$ is assumed to vanish throughout. The total momentum density and density (which follows from similar calculations) are therefore

$$\mathbf{g}(\mathbf{r}) = \mathbf{g}^c + \frac{1}{2}\nabla \times \boldsymbol{\ell}, \quad \rho(\mathbf{r}) = \sum_{\alpha\mu} m^{\alpha\mu}\delta(\mathbf{r} - \mathbf{r}^{\alpha\mu}). \tag{9}$$

The resulting Hamiltonian is

$$H = \int d\mathbf{r}\frac{\mathbf{g}\cdot\mathbf{g}}{2\rho} = \int d\mathbf{r}\left[\frac{(\mathbf{g}^c)^2}{2\rho} + \boldsymbol{\ell}\cdot\boldsymbol{\omega}^c\right], \tag{10}$$

where $\boldsymbol{\omega}^c = \frac{1}{2}\nabla \times \mathbf{v}^c$ is half the CM vorticity of the fluid, the second term was obtained via integration by parts, and a quadratic derivative term was discarded as it will vanish in the long-wavelength limit. The coupling between the vorticity and the angular momentum is key and directly leads to the odd viscosity. This term was present in the previous hydrodynamic derivation of the odd viscosity, but in that reference it was added by hand and with the opposite sign [4].

The dynamics of the momentum density and angular momentum density can be obtained via the Poisson bracket approach to Hamiltonian dynamics after lengthy calculations. The resulting non-dissipative equations of motion for the momentum density are

$$\dot{g}_i + \nabla_j(v_j g_i) = \nabla_j(\eta^o_{ijkl}u_{kl}), \quad \eta^0_{ijkl} = -\frac{1}{4}\ell_n\left(\epsilon_{jln}\delta_{ik} + \epsilon_{iln}\delta_{jk} + \epsilon_{ikn}\delta_{jl} + \epsilon_{jkn}\delta_{il}\right). \tag{11}$$

Specializing to two dimensions, the odd viscosity that appears here is identical to that found in the variational approach if we identify the coefficients $\eta^0 = -\ell_z/2$ and assume isotropy. There are also dissipative contributions to the viscosity that modify the stress tensor to be

$$\sigma_{ij} = -P\delta_{ij} + (\eta^e_{ijkl} + \eta^o_{ijkl})u_{kl}, \tag{12}$$

where P and η^e are the hydrostatic pressure and conventional even viscosity. The dissipative terms arise from the many microscopic degrees of freedom that the coarse-graining procedure followed in the Poisson bracket calculation neglects. In addition, it is possible for dissipative terms related to the angular momentum to appear, but calculating such terms would require keeping track of higher derivative terms in the Hamiltonian. The equation of motion for the angular momentum density is

$$\dot{\ell}_i(\mathbf{r}) + \nabla_j(\ell_i v_j) = \epsilon_{ijk}\omega_j\ell_k - \Gamma(\Omega_i - \omega_i) + \tau_i. \tag{13}$$

Injection of angular momentum is accounted for by τ_i and is balanced by the presence of a finite Γ, which causes dissipation of angular momentum. In the hydrodynamic limit, the resulting steady state has angular momentum $\boldsymbol{\ell} = \mathbf{I} \cdot \tau/\Gamma$, where $\mathbf{I}$ is the moment of inertia tensor of the molecules. These equations of motion are equivalent to those derived in the previous section.

CONSEQUENCES OF ODD VISCOSITY

In this section I will discuss a variety of ways in which the odd viscosity impacts the phenomenology of fluids. In addition to the usual dimensionless parameters used to characterize flows, the Reynolds number $\mathrm{Re} \equiv v_0 r_0/\nu$ and the Mach number $\mathrm{Ma} \equiv v_0/c$, we will also need the odd Reynolds number to characterize the strength of the odd viscosity, $\mathrm{Re}^o \equiv v_0 r_0/\nu^0$. The definitions are constructed from the characteristic velocity and length scales of the initial flow, v_0 and r_0 respectively. and the speed of sound c.

One phenomenon often studied in active matter is vortices. In normal viscous fluids, the inertia of the rotating flow causes a pressure dip at the vortex center, resulting in a decreased density. When the odd viscosity is present there is instead an either increase or larger decrease in the core density, depending on whether the rotation of the vortex and the intrinsic rotation responsible for the odd viscosity align or anti-align. The density deviation profile is plotted in Fig 1a for the case of large odd viscosity, characterize by $\mathrm{Re}^o \ll 1$. The deviation of the vortex core density occurs because the odd viscosity turns rotational motion into radial motion (and vice versa). In fact, the odd viscosity can be absorbed into the pressure in the equations of motion for incompressible flow, $p \to p - 2\eta^o\omega$, because it converts the vorticity into compression or expansion of the fluid.

Another ubiquitous phenomenon in viscous fluids is shocks. Shocks form when a portion of a fluid with higher velocity overtake a potion of the fluid with lower velocity. A simple model capturing this effect is the "half" wave equation $(\partial_t + u\partial_x)u = 0$, where the wave amplitude is the also the propagation speed [7]. Given a Gaussian initial distribution, the peak of the wave will propagate fastest and eventually lead to a multivalued function that looks like a wave crest. Multivalued functions are generally frowned upon by physics, and are avoided by the inclusion of higher-order terms, such as in the Burgers equation, $(\partial_t + u\partial_x)u = \nu\partial_{xx}^2 u$, where ν is the viscosity. These higher order terms not only smooth out the discontinuity, but also determine the propagation speed of the shock. In fluids with odd viscosity, the shock is accompanied by a transverse flow perpendicular to the propagation of the shock. This flow is the result of collisions between the fast- and slow-moving particles converting intrinsic angular momentum into orbital angular momentum. The transverse flow for an ideal Burgers shock with a high odd Reynolds number is shown in Fig. 1b.

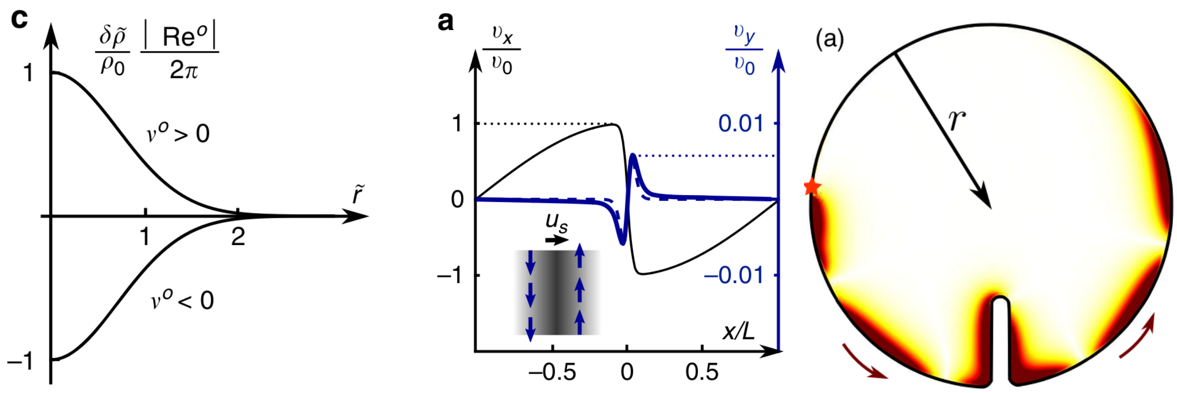
In fluids with a spectral gap, usually induced by Lorentz or Coriolis forces, the odd viscosity leads to an effect familiar in condensed matter physics: topologically protected edge states [8]. The plane-wave solutions of the two-dimensional odd Navier-Stokes equations with a body force term $\omega_B \mathbf{v} \times \hat{z}$ have the dispersion relation

$$\omega_\pm = \pm\omega_B \sqrt{(1 - m\bar{q}^2)^2 + \bar{q}^2}, \quad \bar{q} \equiv |\mathbf{q}|c/\omega_B, \quad m \equiv \omega_B \nu^o / c^2 \tag{14}$$

in addition to a flat band with $\omega(\mathbf{q}) = 0$. In analogy to quantum mechanical plane-waves, a Berry curvature can be constructed from the eigenvectors of the Navier-Stokes equations. In condensed matter physics the Berry curvature is defined on a torus, and the integral of the Berry curvature over this compact space defines a topological invariant [9, 10]. In the continuum case, the parameter space is usually non-compact, but the odd viscosity provides a natural length scale for a UV cutoff, with which the parameter space can be compactified. On this compactified momentum space, the lower and upper bands have Chern numbers $C_\mp = \pm\text{sign}(\nu^o) \pm \text{sign}(\omega_B)$, and the flat band has a vanishing Chern number. According to the bulk-boundary correspondence [9], the fluid will support two co-propagating modes that are protected against back-scattering when $C_- = 2$ and two counter-propagating unprotected edge modes when $C_- = 0$. This was confirmed numerically, as shown in Fig. 1c. Videos of the fluid dynamics for the trivial case, $C_- = 0$, shows two modes propagating away from the source, one in each direction. The counter-clockwise propagating mode scatters off the bottom corner and sets up a standing wave, demonstrating the lack of topological backscattering protection.

EXPERIMENTAL VERIFICATIONS OF ODD VISCOSITY

While the vortex and shock wave physics predicted for odd fluids have so far eluded detection, multiple recent experiments have confirmed the topological edge flows discussed in the previous section. One experiment constructed an odd fluid out of small toys called Hexbugs [11]. The Hexbugs are small rectangular objects equipped with a vibrating motor and special legs that convert the vibrations into linear motion. Each particle of the fluid

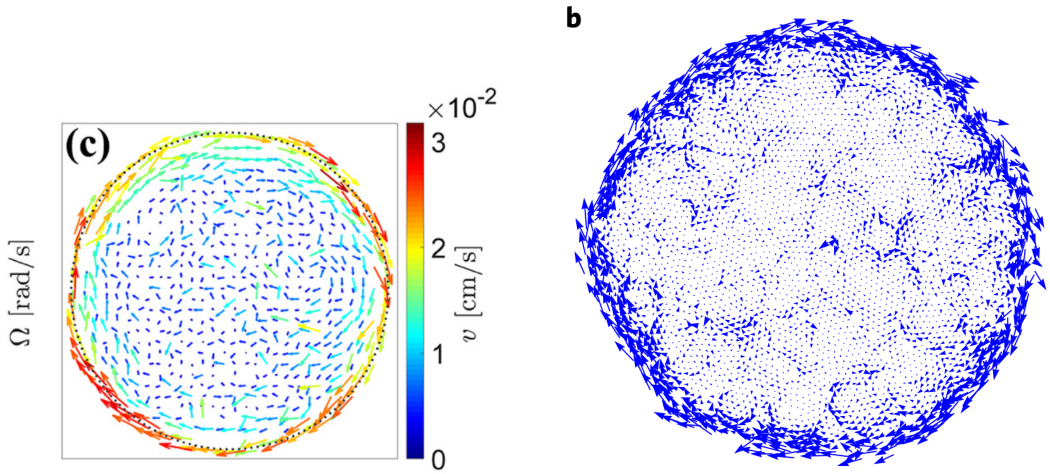


(a) (b) (c)

FIG. 1. (a) The odd viscosity either increases or decreases the fluid density in the core of a vortex depending on the sign of the viscosity and the handedness of the vortex. Plotted here are the rescaled density deviation of a vortex core for positive and negative odd viscosity [4]. (b) Fluid shocks are accompanied by transverse flow in the presence of odd viscosity. The black line is the velocity profile of a Burgers' shock, and the dotted and solid blue lines are the analytical and numerical transverse flows for large odd Reynolds number. The inset shows v_x and v_y in gray shading and blue arrows schematically, respectively [4]. (c) The topologically protected edge mode of an odd fluid with $C_- = 2$. The mode is excited by a sinusoidal source at the star and propagates around the edge of the system. The mode does not back scatter at the bottom corner and decays exponentially into the bulk.

consists of a foam disk with two opposite-facing Hexbugs attached to one side, offset from the center. These particles perform random walks and rotate at $\Omega_0 \approx 8.4$rad/s. This is implementation of an odd fluid is particularly interesting because it allows an observer to see collisions exchange rotation and linear motion with their own eyes, the very mechanism underlying odd viscosity. Another experiment created an odd fluid by suspending millions of micron-scale cubic permanent magnets in water [12]. The magnets are induced to spin at a uniform rate by an external magnetic field. The resulting colloidal fluid demonstrates many features of Newtonian fluids: droplets coalesce, voids collapse like bubbles, and thin regions are unstable to droplet formation.

Despite the linear sizes of the particles in these two experiments differing by six orders of magnitude, both groups are able to identify and track instantaneous particle positions. Plots of particle velocity of both fluids in a circular geometry (actively confined for the hexbug



(a) (b)

FIG. 2. (a) A snapshot of the velocities of individual hexbug particles. The average velocity in the bulk is zero, but there is a strong clockwise flow around the boundary. This boundary flow is robust and appeared in a variety of confining geometries. (b) A snapshot of particle velocities of individual magnets in a droplet. As with the hexbugs, the flow is confined to the edge and is characterized by a penetration depth.

fluid and a naturally formed droplet for the colloidal magnet fluid) clearly demonstrate the robust surface flows predicted in the previous section, as shown in Fig 2. The large linear friction coefficient of the hexbug fluid precludes the measurement of its odd viscosity, but the colloid magnetic experiment provides one of the first precise measurements of odd viscosity. The shear η and rotational viscosity η_{R} are extracted by fitting the penetration depth of the droplet surface flow to theory, and the linear surface friction coefficient Γ_u is determined by measuring the sedimentation rate of droplets on an inclined slide. These coefficients are used to predict the dissipation rate of surface waves on a droplet, which is measured experimentally as the broadening of the power spectrum of waves in a droplet. For a fluid with no odd viscosity, the theory fails to match the measured dissipation rate, but the inclusion of odd viscosity in the theory produces close agreement to the experiment. By fitting the theory to experiment, the odd viscosity is determined with no further fitting parameters, a remarkable feat. This experiment lays the groundwork for further study and engineering of colloidal chiral fluids.

CONCLUSION

Chiral active matter is an rapidly progressing and exciting field. The phenomenology of active fluids with odd viscosity is rich and has deep connections to topology and condensed matter physics. Much work is being done developing and understanding hydrodynamic theories of chiral active matter and recent progress in understanding the microscopic origins of odd viscosity bodes well for the engineering fluids with odd viscosity. As the theory and experimental techniques continue to develop, I expect to see a fruitful back and forth of theoretical predictions and experimental observations that hopefully will lead to transformational new technologies utilizing chiral active matter.

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