Using entanglement to characterize topological phases of matter

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May 13, 2021

Abstract

Off diagonal long range order is often used to characterize emergent phenomena like ferromagnetism and superfluidity. Similarly entanglement entropy and entanglement spectrum can be used to characterize topological phases of matter. In such phases the interplay between interactions, symmetry and topology lead to emergent fractional charge, fractional statistics and non-trivial edge states. In this essay we review how entanglement signatures can be used to deduce these properties from the ground state of the system alone.

1 No country for ODLRO

According to Landau theory, phases like ferromagnets and superfluids are broken symmetry ground states of certain models. The spontaneous breaking of symmetry endows these phases with order, meaning that the spins or order parameter phases are correlated over long distances. This is quantified using the single particle density matrix:

$$\rho(\mathbf{r}, \mathbf{r'}) = \left\langle \psi^{\dagger}(\mathbf{r})\psi(\mathbf{r}) \right\rangle, \tag{1}$$

where ψ represents spin and field operators in models of ferromagnetism and superfluidity respectively. It can be shown that in the ordered phase

$$\lim_{|\mathbf{r}-\mathbf{r'}|\to\infty} \rho(\mathbf{r}, \mathbf{r'}) \neq 0. \tag{2}$$

This property is called off diagonal long range order (ODLRO) and is a signature for ordered phases of matter and the spectacular emergent properties they come with.

However, topological phases like quantum Hall fluids and the SPT phases are not a consequence of spontaneously broken symmetry and do not exhibit long range order in the usual sense. Although it is possible to define a density matrix that does display ODLRO for quantum Hall fluids, we must first map our fermionic model to a system of bosons via a singular transformation[1]. This is not a natural construction and furthermore it fails for SPT phases. Therefore, it is desirable to come up with a property analogous to ODLRO that allows us to characterize topological phases. This property turns out to be entanglement.

We begin by describing briefly how entanglement is quantified in many body systems. We then summarize how it helps us identify topologically ordered and SPT phases and end with a discussion of how these analogues of ODLRO might be measured in the lab.

2 Entanglement

Consider the ground state of some model, $|\Psi\rangle$. For the purposes of this essay $|\Psi\rangle$ is going to represent a phase of matter. In order to understand the entanglement structure in the phase, we construct the corresponding density matrix:

$$\hat{\rho} = |\Psi\rangle \langle\Psi|. \tag{3}$$

Suppose that the Hilbert space of the theory can be partitioned into a tensor product: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We now focus on subsystem A and trace out subsystem B to get the reduced density matrix:

$$\hat{\rho}_A \equiv \mathrm{tr}_B \left(|\Psi\rangle \langle\Psi| \right). \tag{4}$$

We can infer from $\hat{\rho}_A$ whether $|\Psi\rangle$ is entangled or not. If $|\Psi\rangle$ can be written as a product state: $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle$ and $|\psi_B\rangle$ belong in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, it can be shown that $\hat{\rho}_A$ represents a pure state in the subsystem A, meaning it has the property $\mathrm{tr}\left(\hat{\rho}_A^2\right) = 1$. If instead $|\Psi\rangle$ is an entangled state, $\hat{\rho}_A$ represents a mixed state: $\mathrm{tr}\left(\hat{\rho}_A^2\right) < 1$.

From this reduced density matrix we can construct two quantities that allow us to tell apart topological phases from trivial ones.

2.1 Entanglement entropy

We define the von Neumann entropy:

$$S(\hat{\rho}_A) = -\mathrm{tr}\left(\hat{\rho}_A \log \hat{\rho}_A\right). \tag{5}$$

It can be shown that $S = 0$ if $|\Psi\rangle$ is a product state and $S > 0$ otherwise.

Let us now specialize to the case where $|\Psi\rangle$ represents the ground state of a gapped many-body Hamiltonian with only local interactions. We know that for gapped systems, correlations decay exponentially:

$$\langle\Psi| \hat{O}(\mathbf{r}_1)\, O(\mathbf{r}_2)\, |\Psi\rangle \sim e^{-|\mathbf{r}_1 - \mathbf{r}_2|/\xi}, \tag{6}$$

where ξ is the correlation length. If we now physically partition the system into A and B with a boundary[i], $\partial A \gg \xi$, locality ensures that a particle deep in A is unlikely to be correlated to a particle deep in B. Instead the only correlation and, therefore, entanglement between A and B is going to involve a narrow strip around the boundary[2] (see Fig. 1). This leads us to an area law for the entanglement entropy:

$$S \sim \partial A. \tag{7}$$

Kitaev and Preskill postulate the addition of a constant term to this entropy:

[i] In (2+1)D the boundary is a perimeter, $\partial A \gg \xi$. In (3+1)D the boundary is a surface area, $\partial A \gg \xi^2$.

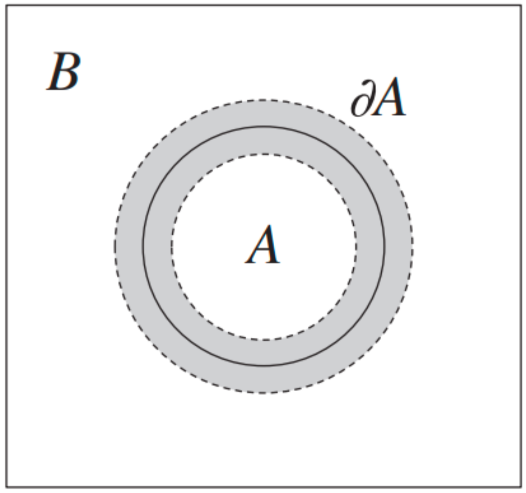

Figure 1: Area law for entanglement in gapped, local theories. Only the shaded region entangles A and B.

$$S = \alpha\,\partial A - \gamma + \mathcal{O}(1/\partial A), \tag{8}$$

where α is a constant that depends on the nature of interactions in the theory and $\gamma \geq 0$ is a constant reduction to the entropy independent of ∂A and, therefore, independent of how we partition the system. If it is non-zero we must conclude that there is long range entanglement in the system since the first term takes into account all short range entanglement between A and B and the last term vanishes in the thermodynamic limit.

For a given state $|\Psi\rangle$ we would like to extract γ. However, this is difficult to do since γ is the subleading term in the expansion of the entropy in powers of ∂A and so we cannot isolate it by taking the $\partial A \to \infty$ limit. Various schemes are used to extract γ in analytical and computational calculations. One approach, used by Kitaev and Levin, involves partitioning the subsystem A into further pieces and summing over entanglement entropies for different combinations of subsystems in such a way that the $\alpha\,\partial A$ terms cancel. Taking the $\partial A \to \infty$ limit then gives us γ. Another method involves calculating S for a range of large system sizes and then extrapolating the result to a system size of 0. We discuss an example of this latter technique in Sec. 3.3.

2.2 Entanglement spectrum

Instead of calculating the entanglement entropy we can choose to diagonalize $\hat{\rho}_A$ and find the entanglement spectrum[3]. It is often easiest to do this by performing the Schmidt decomposition of $|\Psi\rangle$. To do so we recall that $|\Psi\rangle$ lives in the tensor product space, $\mathcal{H}_A \otimes \mathcal{H}_B$, and can be written as:

$$|\Psi\rangle = \sum_{ij} c_{ij} \, |\psi_{A,i}\rangle \otimes |\psi_{B,j}\rangle \,, \tag{9}$$

where $\{|\psi_{A,i}\rangle\}$ and $\{|\psi_{B,j}\rangle\}$ are orthonormal basis sets for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

We now perform singular value decomposition (SVD) on the matrix with elements c_{ij}:

$$c_{ij} = \sum_k \sigma_k \, U_{ik} \, V^*_{jk}, \tag{10}$$

where σ_k are the singular values satisfying: $\sigma_k \geq 0$ and $\sum_k \sigma_k^2 = 1$, and U, V are matrices with orthonormal columns. We can now write down the Schmidt decomposition:

$$\begin{aligned}
|\Psi\rangle &= \sum_k \sigma_k \left(\sum_i U_{ik} \, |\psi_{A,i}\rangle \right) \otimes \left(\sum_j V^*_{jk} \, |\psi_{B,j}\rangle \right) \\
&= \sum_k \sigma_k \, |\phi_{A,k}\rangle \otimes |\phi_{B,k}\rangle \,,
\end{aligned}$$

where $\{\phi_{A,k}\}$ and $\{\phi_{B,k}\}$ are basis sets for the transformed $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Therefore, $\hat{\rho}_A$ becomes:

$$\hat{\rho}_A = \sum_k \, \sigma_k^2 \, |\phi_{A,k}\rangle \langle \phi_{A,k}| \,, \tag{11}$$

where the Schmidt values, σ_k^2's, form the entanglement spectrum of our phase $|\Psi\rangle$. Since efficient SVD libraries are commonplace this method of computing the entanglement spectrum is very convenient.

3 Topological order

3.1 Quantum Hall effects

Consider a gas of N electrons at low temperatures confined to a two dimensional surface of area, A, and subject to a strong magnetic field, B, applied perpendicular

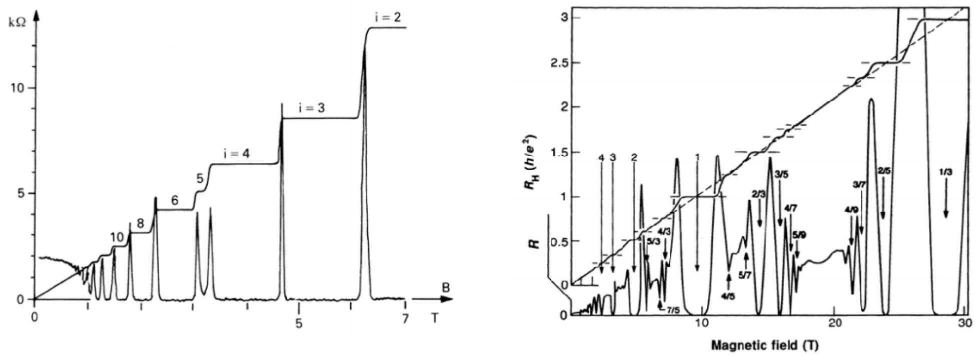

Figure 2: Transverse resistivity against magnetic field in the integer (left) and fractional (right) quantum Hall effects.

to the surface[1]. We can write the quantum Hamiltonian for this system and solve the corresponding eigenvalue problem to find an equally spaced spectrum, called Landau levels. Each Landau level has a degeneracy equal to BA/Φ_0 where $\Phi_0 = 2\pi\hbar/e$ is the flux quantum.

If we now apply a fixed electric field in a certain direction, and measure the resistivity, ρ_{xy}, in the transverse direction while tuning B we find that ρ_{xy} is quantized:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}, \tag{12}$$

where ν is an integer that turns out to be equal to the number of filled Landau levels, $\nu = N/(BA/\Phi_0)$. This is called the integer quantum Hall (IQH) effect and is essentially a consequence of weak disorder in the system. The famous plot displaying the IQH plateaux is shown in Fig. 2.

If we now lower the disorder strength (or equivalently make the magnetic field much stronger), we start to see other values of ρ_{xy} appear. It still obeys the same form as Eq. (12) but now ν takes on rational values. In this context we call ν the filling fraction. For Laughlin states, which we consider in this essay, $\nu = 1/m$ for some integer m, which means that the degeneracy of each Landau level is so large that all our electrons only occupy a fraction of the lowest Landau level. This is called the fractional quantum Hall (FQH) effect and is shown in Fig. 2. Understanding FQH requires taking into account the Coulomb interaction between the electrons.

The most striking aspect of FQH is not quantized resistivity but the discovery, in later experiments, of excitations with fractional charge and statistics. Even though an FQH system is built out of electrons we observe the emergence of charge carriers that are somehow fractions of an electron. An FQH phase with filling fraction, $\nu = 1/m$, has excitations with charge e/m that are neither fermionic nor bosonic. Under an exchange of two such excitations, dubbed anyons, the wavefunction gains a complex phase of $e^{2\pi i/m} \neq \pm 1$. Furthermore, even though FQH phases with different filling fractions have very different excitations and are, therefore, distinct phases, they have the same symmetries. This is unlike any (second order) phase transition studied under Landau's classification and we need a new kind of order with which to classify these phases. This turns out to be topological order.

3.2 Anyons

Topological order and the emergence of anyonic excitations have been studied in other systems as well. For example, certain spin models, called spin liquids also host anyonic particles. In particular, the toric code[2], introduced by Kitaev in the context of topological quantum computing, is the simplest model to exhibit topological order.

The low energy physics of topologically ordered phases is often studied using topological quantum field theories[4] (TQFTs) where the emergent anyons take center stage. Similar to how symmetries determine the appropriate field theory to describe long wavelength behavior of usual phases of matter, the fractional statistic of the anyons determines the appropriate TQFT. This is quantified by the quantum dimension[ii] of the phase, $\mathcal{D}$. For $1/m$ Laughlin states, $\mathcal{D} = \sqrt{m}$, whereas for the toric code, $\mathcal{D} = 2$. Different topological phases can have the same quantum dimension. However, only a trivial phase with no anyons can have $\mathcal{D} = 1$. Therefore, $\mathcal{D}$ allows us to identify topological order by looking at the excitations.

3.3 Topological entropy

While we can identify a topologically order phase by looking at the properties of its excitations, it would be convenient to be able to do so using just the ground state which is often easier to calculate. One way to accomplish this is to look at the eponymous ground state degeneracy of these phases which depends only on the topology of the surface that the model is placed on and is robust against perturbations. In particular, if we place an FQH phase with filling fraction $\nu = 1/m$ on a surface with g genuses, the ground state degeneracy is given by m^g.

[ii] The exact definition of $\mathcal{D}$ comes from the fusion algebra of the anyons, describing which would take us too far afield.

Kitaev and Preskill[5] make the case that it is also possible to identify topological order using just the entanglement properties of the ground state. Specifically they define a topologically ordered phase to be the ground state of a gapped many body Hamiltonian with long range entanglement, captured by a non-zero value of γ in the entanglement entropy. This is a reasonable definition since a non-zero value of γ causes a constant reduction of the entropy, thereby leading to order. Since γ is independent of interactions and partitioning, this order is topological in nature. The authors then relate topological entropy to the quantum dimension:

$$\gamma = \log D. \tag{13}$$

We note that for topologically trivial phases, $D = 1$ and, as expected, we get no long range entanglement.

Kitaev and Preskill prove Eq. (13) by calculating γ from a general TQFT. Levin and Wen[6] come to the same conclusion but via string net condensation. The details of either proof are beyond the scope of this essay. Instead we mention a paper that illustrated Eq. (13) by calculating the ground state entanglement entropy for 1/3 and 1/5 Laughlin states in torus geometry[7]. They numerically calculate the ground state wavefunctions, construct the density matrices and compute the entropy for a range of system sizes denoted by l (see Fig. 3). They show that zero l limit of the entropy is a good approximation for $-\gamma$. In particular the y-intercepts of the 1/3 and 1/5 Laughlin states are 1.13 ± 0.38 and 1.62 ± 0.16 respectively, which are consistent with Eq. (13). Similar calculations have been done for the toric code[2], confirming the validity of Eq. (13).

4 Symmetry protected topological phases

4.1 To be topological or not to be topological

Consider the following one dimensional antiferromagnetic spin-1 chain:

$$H = \sum_i \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + D(S_i^z)^2 \right), \tag{14}$$

where $D \geq 0$ is called the anistropy parameter. The Hamiltonian has time reversal, inversion and translation symmetries. Gu and Wen[8] used numerical methods to show that the spectrum of this Hamiltonian is gapped at all values of D except one. We know that at the gapless point, $D = D_c$, the correlation length will diverge, which is a sign of a second order phase transition. In the Landau paradigm at least one of

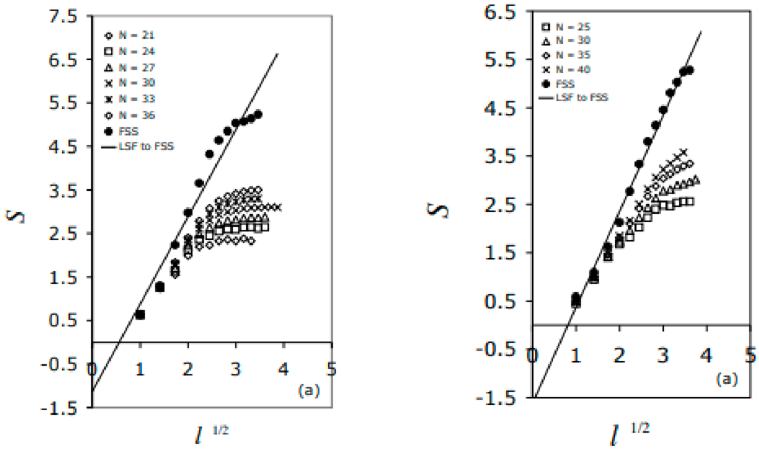

Figure 3: Entanglement entropy with a range of system sizes for 1/3 (left) and 1/5 (right) Laughlin states.

Figure 4: Ground state of the AKLT model

the two phases would be a broken symmetry phase. However, Gu and Wen further show that the ground states on either side of D_c respect all the symmetries of the Hamiltonian.

It is natural to wonder if this is another example of topological order. This seems very likely if we look closely at the ground states. In the large D limit, we have the trivial phase which is simply a tensor product of $|S_z = 0\rangle$ states at each site. However, below D_c, we have the Haldane phase which can be adiabatically connected to the ground state (see Fig. 4) of the AKLT model. The AKLT model has the remarkable property that in open boundary conditions its ground state is 4-fold degenerate and has gapless spin-1/2 edge states. Ground state degeneracies and gapless edge states combined with a gapped bulk are properties observed in the topologically ordered FQH phases we met earlier.

However, while in the FQH phases the ground state degeneracy and gapless edge states are robust against any and all perturbations, in the Haldane phase they are only robust against perturbations that preserve inversion symmetry. However, adding inversion breaking perturbations will gap out the edge and remove the degeneracy. This leads us to conclude that the Haldane phase does not have true topological order. However, it does have topological properties that are protected by certain symmetries and, therefore, we call it a symmetry protected topological phase. In subsequent years many other SPT phases were discovered, including the Kitaev chain which is a 1D superconductor with symmetry protected topological properties, the Chern insulator which is a realization of the IQH phase on a 2D lattice, and topological insulators.

In models with SPT phases, trivial phases can be distinguished from topological ones using certain global order parameters called topological invariants. In the case of the Chern insulator, for instance, one can define a so called Chern number as an integral over the occupied one-particle states. This number is zero in the trivial phase and non-zero in the topological phase. However, this method of characterizing SPT phases becomes very difficult to calculate numerically and are, therefore, impractical for interacting models. Instead we must again look at the entanglement structure of the many body ground state to help us identify topological properties. Note that since SPT phases do not have topological order, by our arguments in the previous section we expect that they do not have long range entanglement and their topological entropy, γ, vanishes. This turns out to be true. However, as we now show, their entanglement spectrum does contain signatures of their topological nature.

4.2 Entanglement spectrum of SPT phases

In Pollmann et al[9] the authors consider a slight generalization of the Hamiltonian in Eq. (15):

$$H = J \sum_i \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{U_{zz}}{J} (S_i^z)^2 + \frac{B_x}{J} S_i^x \right), . \tag{15}$$

In this model we have two additional symmetry broken anti-ferromagnetic phases. These can be described by familiar Landau theory and are not important for the current discussion.

The authors numerically find the ground state of the model for a range of parameters and then use Schmidt decomposition to calculate the entanglement spectrum. The degeneracy in the entanglement spectrum is shown in Fig. 5 where the authors produce colormaps of the difference between the two largest Schmidt values. For the Haldane phase this must always be zero. We see that the Haldane phase exists even in the presence of a perturbation unless that perturbation happens to break inversion symmetry. Note that when the ground state only has inversion symmetry,

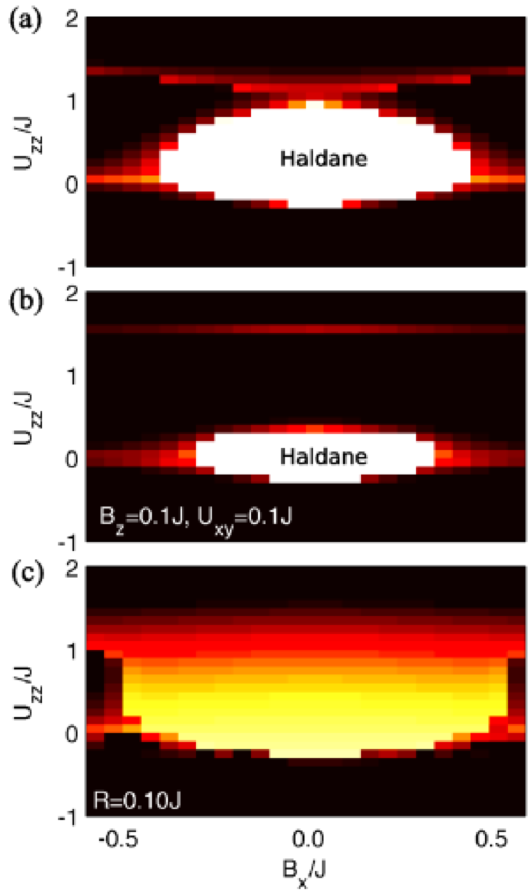

Figure 5: Difference between highest Schmidt values of the ground states of Eq. (15) with (a) no perturbations (b) time reversal breaking perturbations (c) inversion breaking perturbations.

the Haldane phase no longer has gapless edge states. Yet it is an SPT phase in the sense that a symmetry protects it from being adiabatically connected to the trivial phase without going through a gapless point. Therefore, the entanglement spectrum is a superior way to identifying SPT phases than looking for gapless edge states.

This method has been applied successfully to characterize other SPT phases. In fact in a 2010 paper, Fidkowski[10] proves the general result that degeneracies in the entanglement spectrum of non-interacting topological insulators and superconductors correspond to gapless edge states. The essence of the proof seems to be that when we calculate the entanglement spectrum we are required to partition the system and this effectively introduces a boundary. Since gapless edge states are also a consequence of open boundary conditions the entanglement spectrum should contain the same information as the gapless edge states.

5 Measuring entanglement

While it is convenient to calculate entanglement entropy and spectrum numerically, they are very difficult to measure in the lab. Choo et al[11] have shown that it is possible to perform such measurements using quantum computers. They take spin-1/2 chains that are known to have SPT and trivial phases since spin-1/2 degrees of freedom can be conveniently mapped to qubits. They then initialize the qubit in SPT and trivial phases and *measuring* the matrix elements of $\hat{\rho}_A$ by passing the state through a quantum circuit representing $\hat{\rho}_A$ and studying how often certain outputs appear. Although decoherence and noise does affect the data, they are able to see the four-fold degeneracy[iii] in the highest Schmidt value of the SPT phase, and no degeneracy in the case of the trivial phase.

6 Conclusion

In this essay we have sketched the relationship between topological phases and entanglement. This reveals a fascinating application of quantum information to the study of emergence. It would be very exciting to follow this thread further, either by going deeper to gain a more rigorous understanding of the results presented here by studying TQFTs and string net condensation, or by broadening the scope to study how quantum information can shed light on other emergent phenomena, such as

[iii] The experiment works with periodic boundary conditions, which explains the extra factor of 2 in the degeneracy.

biological systems and neural networks.

References

[1] David Tong. Lectures on the quantum Hall effect.

[2] Jiannis K. Pachos. *Introduction to Topological Quantum Computation*. Cambridge University Press, 2012.

[3] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, USA, 10th edition, 2011.

[4] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma. Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, 80:1083–1159, Sep 2008.

[5] Alexei Kitaev and John Preskill. Topological entanglement entropy. *Phys. Rev. Lett.*, 96:110404, Mar 2006.

[6] Michael Levin and Xiao-Gang Wen. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.*, 96:110405, Mar 2006.

[7] B. A. Friedman and G. C. Levine. Topological entropy of realistic quantum hall wave functions. *Phys. Rev. B*, 78:035320, Jul 2008.

[8] Zheng-Cheng Gu and Xiao-Gang Wen. Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order. *Phys. Rev. B*, 80:155131, Oct 2009.

[9] Frank Pollmann, Ari M. Turner, Erez Berg, and Masaki Oshikawa. Entanglement spectrum of a topological phase in one dimension. *Phys. Rev. B*, 81:064439, Feb 2010.

[10] Lukasz Fidkowski. Entanglement spectrum of topological insulators and superconductors. *Phys. Rev. Lett.*, 104:130502, Apr 2010.

[11] Kenny Choo, Curt W. von Keyserlingk, Nicolas Regnault, and Titus Neupert. Measurement of the entanglement spectrum of a symmetry-protected topological state using the IBM quantum computer. *Phys. Rev. Lett.*, 121:086808, Aug 2018.