Network theory applications in complex systems

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Abstract

Network theory provides a great framework to study a myriad of phenomena occurring in nature. In this report we describe the different ways in which networks are characterized by studying empirical networks. We then study the popular descriptive models used to build and study networks with properties similar to empirical networks. We also explore methods from physics which can be used to analyze networks, including statistical mechanics and dynamical systems. Finally, we discuss the potential of renormalization group theory to study networks.

1   Introduction

Complex networks can be used to describe complex systems which can be seen in a wide variety of fields like sociology, mathematics, economics and physics. Essentially any system with free agents or degrees of freedom communicating to each other through what can be approximated as a single channel can be thought of as a network. This kind of formulation is readily applicable to a number of problems, which is why the study of complex networks have applications in various fields.
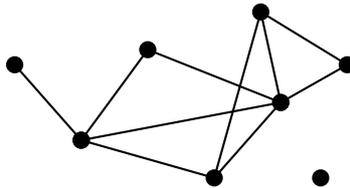
Figure 1: A simple network with 8 nodes and 10 vertices.

A simple network can be formed with nodes connected by binary links, which either exist or don't. More complicated networks can have links which have a number associated with them representing the strength of the link or interaction. Networks can be described using the notation $G(V,E)$ where V is the set of nodes and E is a set of ordered pairs of nodes which have a link connecting them. A popular way of describing a simple network is by using the adjacency matrix associated with the network. The adjacency matrix is defined as:

$$A_{ij} = \begin{cases} 1 & \text{if there is a link between nodes i and j} \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

The entries of the adjacency matrix completely define the network. Simple networks have various observable properties which reveal important information about the network. Some of these are:

- the degree distribution $p(k)$, which is the probability that a randomly chosen has degree k, which means it is linked to k other nodes.

- the average path length, which is the average number of steps between any two nodes in the network.

- the average clustering coefficient, the clustering coefficient for a node i is defined as the number of closed triplet in $\mathcal{N}_i$ divided by all possible triplets in $\mathcal{N}_i$ including node i. $\mathcal{N}_i$ is defined as the neighbourhood of i which consists of nodes which are linked to i.

- the size of the largest connected component.

- the spectral properties of the adjacency matrix.

The types of networks which are dealt with in most of physics have a high degree of symmetry associated with them. For example, the translational symmetry of lattices are what ultimately make them tractable for theoretical analysis. In the case of networks there is no such mathematical structure, which makes their analysis much harder, and so we study networks by looking at universal characteristics such as those defined above.

However, it must be noted that the intractability of the theoretical analysis of networks does not relieve them of emergent phenomena. The development of motifs and patterns

such as community structures, correlations and power-law degree distributions discussed in sections 3, 4 are strong indicators of emergent behaviour. The phenomena of consensus formation and synchronization, studied in section 5, also hint towards emergence.

2   Empirical networks

In this section we consider the various properties observed in real-life networks which as it turns out have very similar properties. For example, consider networks in sociology, which are networks of people with each node representing a person, and a link representing contact. The study of these networks can be used to reveal various aspects of a society like community structures. One particular application is predicting the spread of diseases using social networks in Epidemiology.
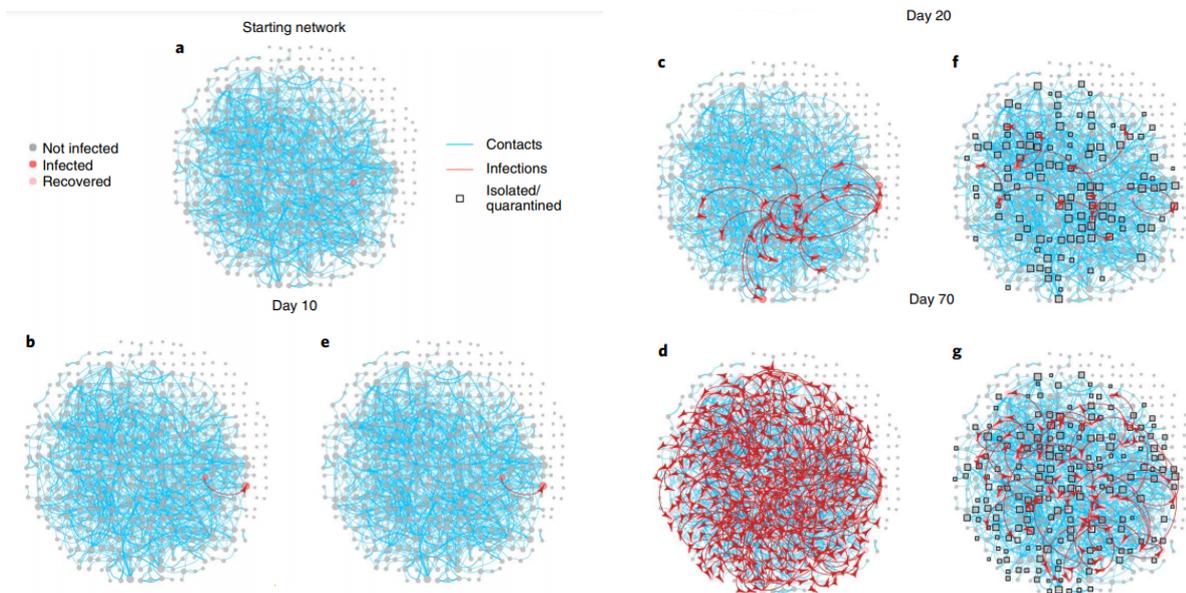
Figure 2: Simulating the spread of Covid-19 from a single infected individual(Image from [1]).

Other great examples include the networks of the world wide web(WWW). In this case a node represents a website, and a link represents a hyperlink reference from one site to the other. In this case the graph is directed.

2.1   Important characteristics of Empirical networks.

Among the various characteristics in these networks some of the more striking ones are that of the small-world phenomena, power law degree distributions and high clustering coefficients.

2.1.1   Small world phenomena

Small-world phenomena describes networks with a small average path length, which is the average number of steps required to move from any given node to another. Another indicator of small-world behaviour would be a small network diameter, which is the largest distance between any two nodes in the network. If the distance between node i and node j is denoted by d_{ij}, then the average path length is given by $\langle L \rangle = \frac{2}{n(n-1)} \sum_{i,j} d_{ij}$ and the network diameter

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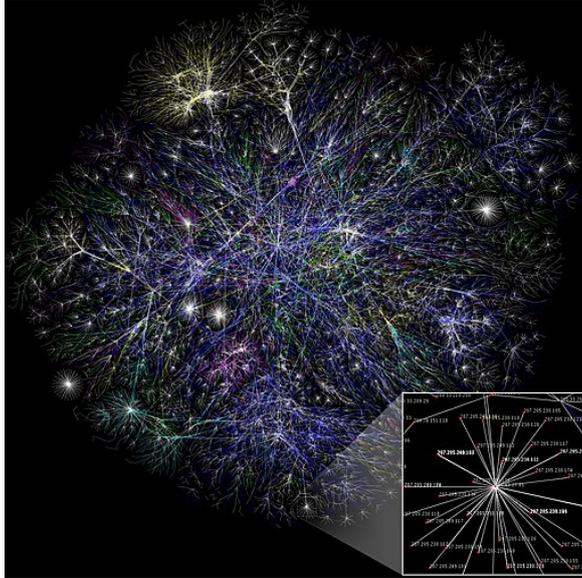
Figure 3: The Opte Project Map of the Internet, Barrett Lyon, 2003. Creative Commons.

is given by $D = \max\{d_{ij}\}$. The network diameter dependence characteristic of small-world behaviour is $D \sim \log(N)$, where N is the number of nodes in the network.

2.1.2 Power law degree distributions

The degree of a node is defined as the number of neighbours it is attached to. It can be calculated using the adjacency matrix using $k_i = \sum_j A_{ij}$. The distribution of node degrees $p(k)$ tells us the fraction of nodes with degree k. The power-law behaviour of node degree implies $p(k) \sim k^{-\gamma}$ where $\gamma > 0$ is the critical exponent. The power-law distribution of the node degrees hints towards scale-free behaviour [2, 3, 4, 5].

Power-law behaviour can be seen clearly in a log-log plot of the data, which has a dominant linear spectrum in case of power-law dependence. For example if $p(k) = p_0 k^{-\gamma}$ then $\log(p(k)) = \log(p_0) - \gamma \log(k)$, which is the linear behaviour we expect.

2.1.3 Clustering coefficient and transitivity

Apart from the above two characteristics a high average clustering coefficient and transitivity is also common in empirical networks [6, 7, 8]. Transitivity simply means that for some a, b, c if $A_{ab} = A_{bc} = 1 \implies A_{ac} = 1$. This is a very common theme in sociology where strong social connections between person A and person B, and person A and person C make it very likely that there is a social connection between person A and person C [8]. A measure of the local clustering in a network is given by

$$C_i = \frac{\text{number of links between the nearest neighbours of node i}}{\text{max number of links between the nearest neighbours of node i}}$$

Since transitivity implies that the number of triangles in a network is high, a high degree of satisfying transitivity also implies a high clustering coefficient. A clustering coefficient around $1/2$ is considered to be high. Networks with a tree like structure, like fractal networks, usually have a low average clustering coefficient.

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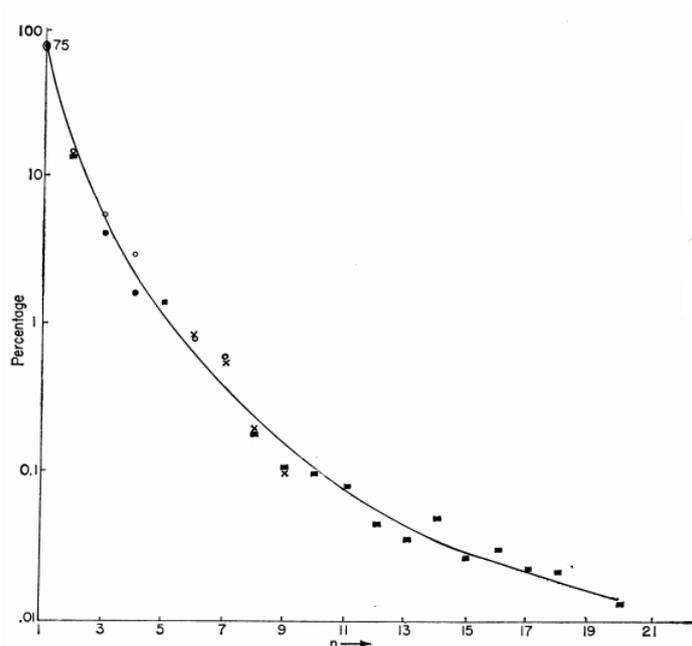
Figure 4: The figure shows the percentage of papers as a function of number of citations demonstrating the scale free behaviours in citation networks of research papers(Image from [3]).

3 Descriptive models for networks

We now discuss the popular theoretical models developed for the study of networks. We will first study the Erdös-Rényi model [9] and discuss its properties. We will see that the model does not have properties similar to the empirical networks discussed in the last section. Subsequent models tried to solve some of the shortcomings of the Erdös-Rényi model. Among these we present the Watts-Strogatz model [7] and the Barabasi-Albert model [10]. All of these models are stochastic models which study ensembles rather than single networks. The advantage when considering an ensemble of networks instead of working with individual networks is that you can calculate analytically some of the important observables related to the classes of networks with relative ease using ensemble averages.

3.1 Erdös-Rényi random network

A random graph is a graph with a given set of nodes with the links connecting them following a given probability distribution. One of the earliest models of a random graph was given by Erdös and Rényi[9], popularly known as the Erdös-Rényi model. The model gives us classes or ensembles of networks denoted as $G(n, p)$ which are classes of networks with n nodes, with a link between each pair of nodes occurring with a probability of p.

It can be easily shown that:

- The average number of edges is : $\dfrac{n(n-1)}{2}p$.

- The degree distribution is given by: $P(k_i = k) = P(k) = {}^{n-1}C_k p^k (1-p)^{n-1-k}$ which in the continuum limit with $n \to \infty$ and fixed $\lambda = pn$ gives us the Poisson distribution $P(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$. This result is at odds with what is seen in Empirical networks.

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(a) $p = 0.3$ (b) $p = 0.6$ (c) $p = 0.9$
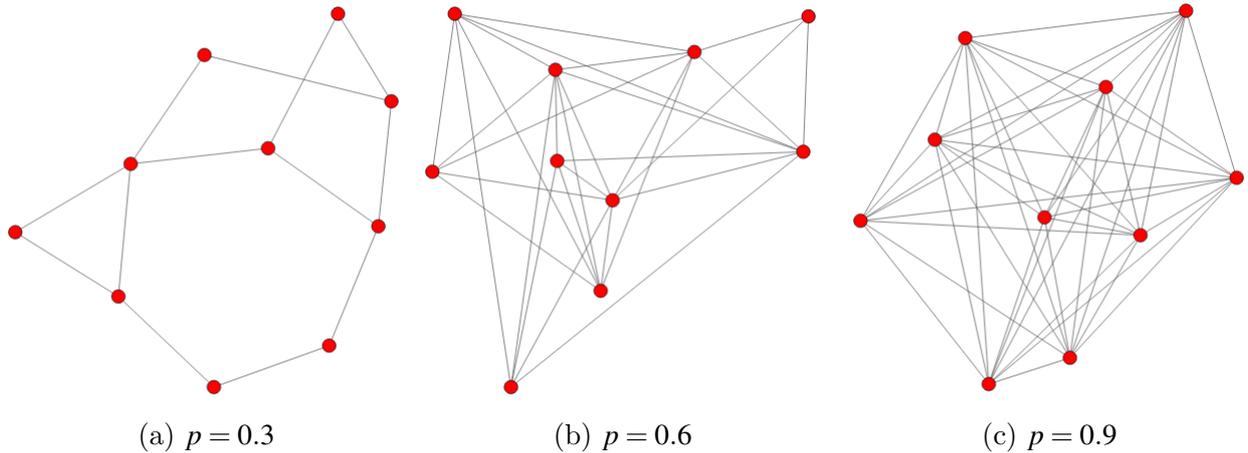
Figure 5: Erdös-Rényi networks with $N = 10$ nodes and different probabilities of linking p.

- Clustering coefficient: $C_i = \dfrac{p\dfrac{k_i(k_i-1)}{2}}{\dfrac{k_i(k_i-1)}{2}} = p$. In the continuum limit, the average clustering coefficient $C = p = \dfrac{\lambda}{n} = 0$. Which is also something which we do not see in empirical networks.

- the average path length can be shown to be $\langle L \rangle = \dfrac{\log(n)}{\log(pn)}$.

3.2 Barabasi-Albert model

The Barabasi-Albert model [10] is a dynamic growth model with the network growing with every time step following a given probabilistic rule. We start at time t=0 with n_0 nodes and some $m_0 \geq n_0$ edges. For each time step, we add an extra node with $m \leq n_0$ links to the earlier network. The links are chosen such that node i has a probability $\Gamma_i = \dfrac{k_i}{\sum_i k_i}$ of linking with the new node. This rule enforces preferential attachment in the network, since nodes with higher node degree have a greater probability of increasing their node degree. Thus at time step t, we have $n_0 + t$ nodes and $m_0 + tm$ links.

We now present the characteristics of the model without proof:

- Node degree distribution follows a power law given by $p(k) = \dfrac{2m}{k^3}$.

- The average path length is given by $\langle L \rangle \sim \dfrac{\log(N)}{\log(\log(N))}$.

- The clustering coefficient is given by $C \sim N^{-\frac{3}{4}}$.

The characteristics of this model are evidently much closer to those of empirical networks when compared to the Erdös-Rényi model. However the Clustering coefficient of the model still tends to zero

$$\lim_{n \to \infty} C \to 0$$

in the limit of large node number which is in contrast to what is observed in empirical networks.

(a) $N = 10, t = 0$ (b) $N = 15, t = 5$ (c) $N = 20, t = 10$
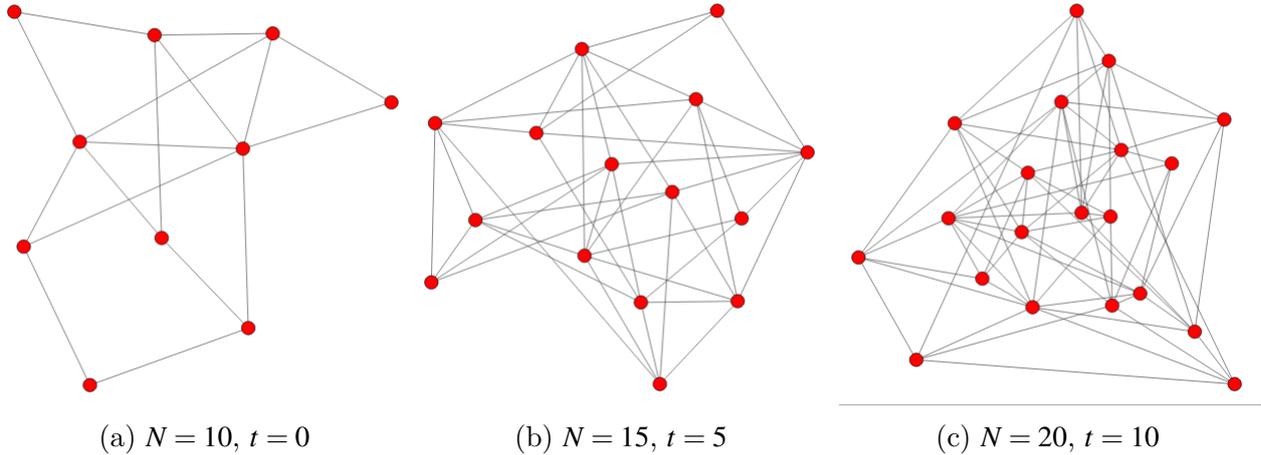
Figure 6: Sequence of Barabasi-Albert graphs with $n_0 = 10$, $m_0 = 17$, $m = 5$.

3.3 Watts-Strogatz model

The Watts-Strogatz model [7] is a model with a single parameter p which decides the randomness of links in the model. We start with a regular lattice with n nodes and an even k edges per node which connect to the k nearest nodes. Then we randomly reconnect each link with a probability p. Clearly in the extreme limit $p = 0$, we have a regular network whereas in the other extreme limit $p = 1$, the network would correspond to a completely random network and hence would be an instance of an Erdös-Rényi graph.
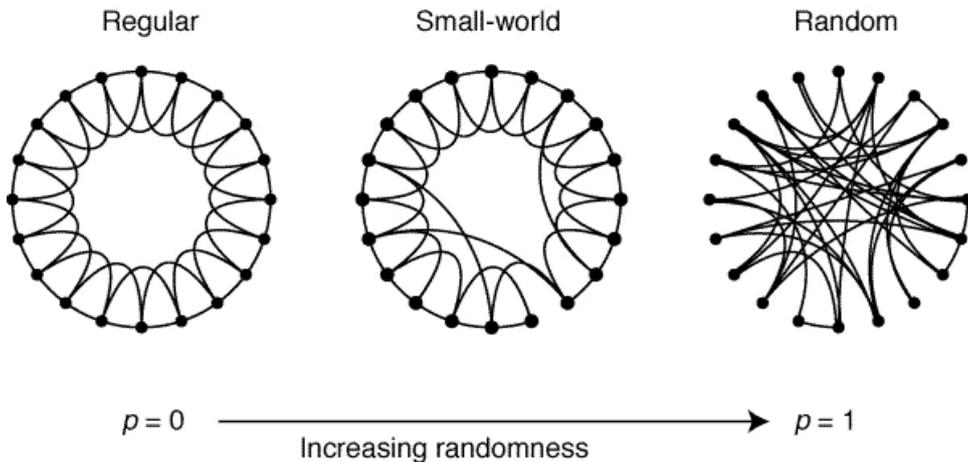
Figure 7: Changes in the Watts-Strogatz network structure with increasing randomness(Image from [7]).

We now discuss the characteristics of the Watts-Strogatz model in the regime $n \gg k \gg \log(n) \gg 1$. Considering the properties in the two extreme regimes we find [7]:

- the node degree distribution is Poisson-like.

- the average path length:

$$\lim_{p \to 0} \langle L \rangle \sim \frac{n}{2k} \gg 1 \qquad\qquad \lim_{p \to 1} \langle L \rangle \sim L_{ER} \sim \frac{\ln(n)}{\ln(k)} \qquad (2)$$

- the clustering coefficient:

$$\lim_{p \to 0} C \sim \frac{3}{4} \qquad\qquad \lim_{p \to 1} C \sim C_{ER} \sim \frac{k}{n} << 1 \tag{3}$$

The above properties suggest that we would encounter networks with characteristics similar to empirical networks somewhere in between the parameter range. This is because we require a high clustering coefficient like in the case with $p = 0$ but at the same time also require long-range re-connections which give us a low average path length like in the case of $p = 1$. We also note that the Poisson like degree distribution is in contrast to the properties of empirical networks.

4 Statistical mechanics of networks

Consider a simple network G, with an adjacency matrix given by A_{ij}. Let each link in the network be weighted by an external parameter β which is an indicator of the external stresses on the network. For example, β could correspond to the levels of a chemical acting on a network of proteins. Another example is the case of network models of business relationships between economic firms where β would correspond to the present state of the stock price. Note that β can also be interpreted as the strength of the links between nodes, with a high β corresponding to a network where the links are dominating over the external stresses of the system.

The Estrada index [11, 12] is then denoted as $EE(G, \beta)$, and is given by the formula:

$$EE(G, \beta) = Z(G, \beta) = Tr(\sum_{r=0}^{\infty} \frac{\beta^r A^r}{r!}) = Tr(e^{\beta A}) = \sum_{i=1}^{n} e^{\beta \lambda_i} \tag{4}$$

where λ_i correspond to the eigenvalues of the adjacency matrix. Note that $(A^k)_{pq}$ corresponds to the number of paths with k steps from node p to node q. We weight each of these paths by a factor $\frac{1}{k!}$ to make shorter paths have a higher weight as compared to longer paths. We now note that the number of self-paths would be a good measure of node centrality, and summing over all such paths for some node i gives us $\sum_k \frac{(A^k)_{ii}}{k!} = Tr(e^A)$, which is identical to the Estrada index. The centrality of a node is a measure of the importance of the node, which takes into account how a node connects to different parts of the graph. The Estrada index is one of the various schemes and types of centrality measures like closeness centrality, betweenness centrality and eigenvector centrality.

The analogy with statistical mechanics can be used to define a configuration/state of the network as the eigenvector of the adjacency matrix given by the eigenvalue λ_j. The probability of being in a configuration j is then given by:

$$p_j = \frac{e^{\beta \lambda_j}}{Z(G, \beta)} \tag{5}$$

Using this we can define various quantities which are analogues of different thermodynamic functions like:

- the entropy for the network

$$S(G, \beta) = -\sum_j p_j \ln(p_j)$$

with a bound given by $0 \leq S(G,\beta) \leq \beta \ln(n)$.

- the total energy for the network

$$H(G,\beta) = -\sum_j \lambda_j p_j$$

with a bound given by $-\beta(n-1) \leq H(G,\beta) \leq 0$.

- the Helmholtz free energy of the network:

$$F(G,\beta) = -\frac{1}{\beta} \ln(Z(G,\beta))$$

with a bound given by $-\beta(n-1) \leq F(G,\beta) \leq -\beta \ln(n)$.

Note that we obtain the bounds by considering the extreme limits for a complete graph as $n \to \infty$ and the null graph, that is a graph without any links.

4.1 Correlations in networks

The analogues of thermodynamic functions defined above can be used to characterize various properties of the network. The simplest example is that of the correlations between two nodes, say i and j, of the network. The correlation, denoted as X_{ij}, can be characterized by the number of pathways information can reach from node i to node j. Weighting each of these paths with a factor depending on the length as done in the last section we obtain:

$$X_{ij} = \sum_{k=1}^{\infty} \frac{(\beta A)_{ij}^k}{k!} = (e^{\beta A})_{ij} = \sum_{l=1}^{\infty} v_l(i)v_l(j)e^{\lambda_l} \tag{6}$$

where v_l is the eigenvector corresponding to the eigenvalue λ_l. This measure of correlations is analogous to the Green functions in statistical mechanics.

5 Dynamical processes on networks

So far we have considered networks without any dynamics, but introducing dynamics in networks is a very natural extension. Dynamical processes in networks are ubiquitous in nature, from the signals in neurons to the interactions of people on wall street to decide the stock price, these systems can very naturally be described as dynamical processes on networks. It is important to note that we do not change the structure of the network, the dynamics is included in the nodes which now hold some information which can be transferred or exchanged through the links. Another minor extension of simple networks is to let the links now have different strengths corresponding to how strongly two nodes are connected or influence each other. Let ζ_i denote the information in each node.

5.1 Consensus

A dynamical process can be defined using the equations:

$$\zeta_i(t+\delta) = \zeta_i(t) + \delta \sum_{\langle i,j \rangle} A_{ij}[\zeta_j(t) - \zeta_i(t)] \tag{7}$$

$$\zeta(0) = \zeta_0 \tag{8}$$

where the first equation governs the evolution and the second equations gives us the initial conditions for the network. δ is the step size in time, and $\langle i,j \rangle$ denotes every pair i,j which are linked.

A simpler way of writing the above equations is:

$$\zeta_i(t+\delta) = \sum_j P_{ij}\zeta_j(t) \qquad\qquad \zeta(0) = \zeta_0 \tag{9}$$

where $P_{ij} = I_{ij} - \delta(diag(k_i)\delta_{ij} - A_{ij})$ is called the Perron matrix for the network.

A consensus is said to be reached when for a given set of initial condition ζ_0, we see $|\zeta_i(t) - \zeta_j(t)| \to 0$ as $t \to \infty$. The parameters of the networks and the dynamics decide if a network can reach consensus. For example, in the case of completely connected simple networks a consensus is reached as long as the eigenvector, say v_i^{max}, corresponding to the largest eigenvalue of the Perron matrix is uniform, ie. $v_i^{max} = v_0 \forall i$. It can be shown that this condition is satisfied when $0 \le \delta \le k_{max}^{-1}$ where $k_{max} = \max_i(k_i)$ is the maximum degree of the network.

5.2   Synchronization

Synchronization is a phenomena that is ubiquitously seen in nature. Systems of coupled oscillators are a common example of networks which attain a synchronous state. Synchronization is not limited to systems with symmetric mathematical structure and can occur in systems which are seemingly random. Like for example, schools of fish and swarms of bees where different agents synchronize their movements with that of other agents. To explore this phenomena we consider networks of coupled oscillators with arbitrary topology. In this section we discuss the conditions under which synchronization takes place following the paper [13].

Consider a network $G(V,E)$ with n nodes. Let a given node i corresponds to a dynamical system, represented by the vector $x_j^i, j = 1,,,,,l$. The topology of the graph is encoded in the Laplacian matrix for the network given by $L_{ij} = diag(k_i)\delta_{ij} - A_{ij}$. The most general formulation for the equation of motions read:

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^n L_{ij}H(x_j) \tag{10}$$

where $x_i \in \mathbb{R}^l$ represents the state of node i, σ is a parameter which represents the degree of coupling and $f : \mathbb{R}^l \to \mathbb{R}^l$ and $H : \mathbb{R}^l \to \mathbb{R}^l$ encode the dynamics of the system. H is called the outer coupling matrix. The network achieves synchronization if the system reaches a steady state such that $x_i(t) = s(t) \forall i$ as $t \to \infty$.

The stability analysis of the steady state solution, if it exists, gives us the required condition for the existence of the solution. Consider a small perturbation to the steady state given by, $x_i = s + \delta_i$. Expanding about the steady state solution:

$$f(x_i) = f(s) + \delta_i f'(s) \tag{11}$$

$$H(x_i) = H(s) + \delta_i H'(s) \tag{12}$$

The evolution of the deviations from the steady state solution is given by:

$$\dot{\delta}_i = f'(s)\delta_i + \sigma \sum_{j=1}^{n} [L_{ij} H'(s)]\delta_j \tag{13}$$

The system of equations can be decoupled by using the eigenspectrum of the Laplacian matrix. Let λ_i and ϕ_i be the eigenvalues and eigenvectors of the Laplacian matrix, ie. $L_{ij}\phi_j = \lambda_i \phi_i$. Using which we have:

$$\dot{\phi}_i = [f'(s) + \sigma \lambda_i H'(s)]\phi_i \tag{14}$$

Assuming that the time-scales of variation in the steady state solution are large enough for us to solve the above equations, we have:

$$\phi_i(t) = \phi_i(0) \exp\{[f'(s) + \sigma \lambda_i H'(s)]t\} \tag{15}$$

For stability the factor in the exponential should be negative for all i, ie. $f'(s) + \sigma \lambda_i H'(s) < 0 \forall i$. To find the regime of stability for a given problem one should plot the master function $M(\alpha) = f'(s) + \alpha H'(s)$ for the typical functions f, H over their relevant parameter spaces. Carrying out such a procedure will give you a plot similar to Fig. 8 with critical values of the parameter α denoted as α_1, α_2.
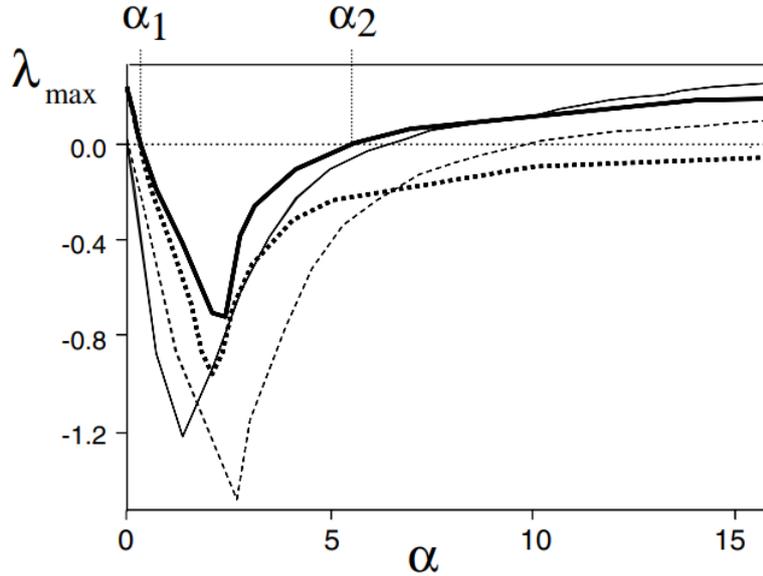
Figure 8: Critical values of the parameter α for the problem of Rössler oscillators(Image from [13]).

The condition for stability then reduces to:

$$\frac{\lambda_{max}}{\lambda_1} \leq \frac{\alpha_2}{\alpha_1} \tag{16}$$

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where λ_{max} is the largest eigenvalue and λ_1 is the first non-zero eigenvalue of the Laplacian matrix for the network. Hence, we conclude that a small ratio of $\frac{\lambda_{max}}{\lambda_1}$ favours the synchronizability of a network, and is solely decided by the topology of the network.

6   Ideas of Renormalization group flows on networks

In this section we use renormalization group(RG) methods to distinguish between networks with fractal characters and networks with small-world properties, following the paper [14]. Note that a fractal network is defined as a network which has the same degree distribution as that of the coarse-grained network, which we do by using the box covering algorithm. The box-covering algorithm is well defined [15] for most networks, so let us assume such a coarse-graining procedure exists. Since, coarse-graining does not change the degree distribution of fractal networks, they cannot have the long-range shortcuts required by the small-world phenomena. This is made apparent by the fact that for small-world networks we require the diameter to grow slowly $D \sim \log(N)$, whereas in the case of fractal networks we would expect it to grow as a power law.

Consider an underlying fractal network, denoted as G_0, on top of which we add long-distance shortcuts following a probability distribution $p(r) = Ar^{-\alpha}$. The final network with the shortcuts is denotes by G. Next, we consider the coarse-graining of this network using the box-covering approach. Let the length scale associated with the coarse-graining be b, which is the maximum distance between nodes in each box. Note that the box-dimension is given by $\frac{N}{N_b} = b^{d_b}$, where d_b is defined as the box dimension. The coarse-grained network is denoted as $G_b = R_b(G)$. Since, the underlying network is fractal we have $R_b(G_0) = G_0$ and hence it is necessary to only look at the change in the shortcuts to understand the RG-flow.

The renormalized probability, denoted $p_b(r)$, for shortcuts is given by:

$$p_b(r) = 1 - P(\text{no shortcuts at distance r in } G_b) \tag{17}$$

No shortcuts at distance r in the network G_b implies there are no shortcut at distance br among the b^{d_b} nodes in the network G which correspond to the representative averaged nodes in G_b, the probability of which is given by $(1 - p(br))^{b^{d_b} * b^{d_b}}$. In the limit $b \to \infty$ this reduces to $\lim_{b \to \infty} p_b(r) = 1 - \exp\left(-Ar^{-\alpha}b^{-\alpha+2d_b}\right)$.

The above analysis gives us a critical point $\frac{\alpha}{2d_b} = 1$. For $\frac{\alpha}{2d_b} < 1$ we have a regime where $p_b(r) \to 1$ which gives us a fully connected network. For $\frac{\alpha}{2d_b} > 1$ we have a regime where $p_b(r) \to 0$, which is a regime where the shortcuts disappear at large distances and we are left with the underlying fractal structure. As for the critical value $\alpha = 2d_b$ we obtain an RG-flow towards a fractal network with added shortcuts governed by the probability distribution $p(r) = 1 - \exp\left(-Ar^{-2d_b}\right) \xrightarrow[r \to \infty]{} Ar^{-2d_b}$, which is qualitatively the same picture we started with but with renormalized parameters.

7   Conclusion

Network science is a rich-avenue for research into complex systems. A number of complex systems have a natural descriptions in terms of networks. We studied some of the common

properties of networks seen in nature and the descriptive models which have been developed to reproduce these characteristics. A wealth of interesting phenomena can be modelled as dynamical processes on networks. There is great potential to use methods from physics to classify and understand the properties of networks.

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