Anderson Localization

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Abstract

This work covers the theory of Anderson localization. The behavior of Anderson's original model is discussed, as is the derivation of Wegner's nonlinear σ model. A relationship to Goldstone's theorem in this context is covered. We then discuss an experimental demonstration of Anderson localization in a three-dimensional ultracold atom system. Finally, extensions to weak interactions via theory and numerics are discussed.

Contents

1 Introduction

In physics, we often study systems through the lens of symmetry. Symmetries enforce structure and regularity, often allowing for especially elegant analyses and results. Furthermore, highly symmetric systems typically have properties termed *universal*: Dictated by high-level structure rather than the more complicated microscopic physics. Disordered systems are in some sense the opposite of highly symmetric systems. Instead of rigidly enforced structure, they have a strictly-specified lack of structure. Despite this, disordered systems have an elegance of their own. Universality itself, in fact, is found in classes of disordered systems whose large-scale or average-case physics turns out to be independent of the details of the distribution from which the system is drawn. Some of these universal properties are quite counterintuitive, at least for intuitions developed on non-random systems.

In a periodic potential, there are no stable localized states. Any initially localized particle will diffuse out over the scale of the whole system. This is true both for a quantum particle and for a classical particle subject to thermal fluctuations. In a potential with random, bounded spatial variation, thermal fluctuations will still drive a classical particle arbitrarily far. However, the same is not always true for a quantum particle. Instead, there sometimes exist localized energy eigenstates which remain in some region much smaller than the scale of the system for all time. A potential which is statistically translationally-invariant (that is, drawn from a translationally-invariant distribution) gives qualitatively very different behavior than a potential with exact translational invariance.

Localization has a few interesting physical consequences. First, the effects of quantum fluctuations are qualitatively different from those of classical thermal fluctuations in this setting. This implies a failure of ergodicity, since in the absence of an external heat bath the system will "remember" its initial state indefinitely. Furthermore, since a localized system doesn't have long-range transport, conventional thermodynamic assumptions about extended systems acting as heat baths for themselves will be violated. [1] This absence of long-range transport also leads to the destruction of superconductivity when sufficient disorder is introduced into otherwise-superconducting materials. [5] The rest of this work will be devoted to studying the origin and dynamics of localization.

2 Anderson's approach

The theory of localization was first worked out by Anderson in 1958. [2] Consider a tight-binding model in three dimensions, with a Hamiltonian consisting of an onsite term E_j and some hopping terms V_{jk}:

$$H = \sum_j E_j c_j^\dagger c_j + \sum_{j,k} V_{jk}(c_j^\dagger c_k + \text{h.c.})$$

We assume that V_{jk} is fixed and depends only on the distance between sites j and k. We also restrict V to be short-ranged (to decay at least as fast as $|r_{jk}|^{-(3+\epsilon)}$,

in particular). Beyond this restriction, the only feature of V_{jk} which turns out to be important is its overall scale V. The onsite potentials E_j, on the other hand, are random. For simplicity, we take them to be i.i.d. uniform on $[-W, W]$, where the parameter W controls the strength of the disorder. The details of this distribution are not important; the key point is that it has characteristic spread W. The dimensionless ratio $\frac{W}{V}$ is then the meaningful measure of the strength of the disorder.

To study localization, Anderson begins with a state which is entirely localized on one site and studies the long-time evolution. In a diffusive system, the amplitude for the particle to remain at the initial site should decay to zero for long times. The key finding is that there exists some critical value of $\frac{W}{V}$ below which the amplitude is instead finite even as $t \to \infty$. This indicates the existence of localized states which do not diffuse away.

The strategy used to obtain this result requires first assuming that single hops from the inital site dominate (i.e. V_{0k} for an initial site 0), then accounting for the influence of other V_{jk} perturbatively. A key intuition is that transitions mostly happen between pairs of sites for which $\frac{E_j - E_k}{V_{jk}}$ is small. Stronger disorder makes it less likely that any given E_j and E_k are close together. If V_{jk} is short-range and W is large, then with high probability most sites won't have any partner within hopping distance with sufficiently similar energy. However, if V_{jk} is very long range, then there are many chances to find some E_j very close to E_k just by luck, and so transport is possible. This description is reminiscent of a percolation transition, where states are localized if and only if they are trapped in an "island" of sites with no strong external connections.

To create a more tractable special case, we can assume that each site is connected to exactly Z other sites, all with equal weight. As in a percolation transition, one can define a connectivity parameter which quantifies how connected sites are to distant parts of the lattice. More precisely, we count the number of acyclic paths of length L starting from a site. For large L, we expect this quantity to be $O(K^L)$ for some constant K. K is then our connectivity. It is then possible to estimate the critical disorder strength needed to produce localization as a function of K. Anderson's numerical estimates and analytic upper bound are shown in figure 1. These results are consistent with the heuristic picture presented above.

3 Field-theoretic models

Another approach is that described by McKane and Stone in [**mckane˙stone**]. The goal is to describe localization in terms an effective field theory, known as Wegner's nonlinear σ model.

We start with solutions of Schrödinger's equation on a continuous space with a random Gaussian noise potential. The time-evolution of a maximally localized initial state is given by the retarded Green's function $G(x, y, t, V)$. If at long times G is nonzero for any fixed x, y, then localized states exist. However, more interesting results can be obtained by picking out the components of the initial

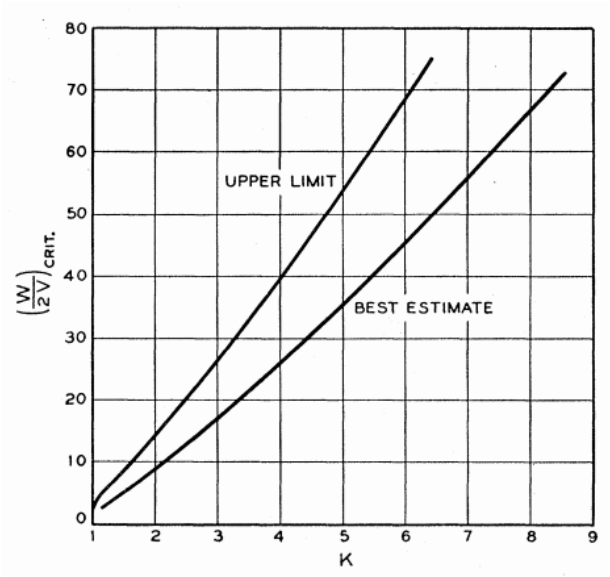

Figure 1: From [2]. Numerical estimates for the critical disorder strength, plotted against the connectivity K. We see that systems with strong connectivity can support diffusion at higher levels of disorder.

state at particular energies using

$$G(x, y, E, V) = \int_0^\infty e^{iE} G(x, y, t, V) t d$$

This is the Green's function of the time-independent Schrödinger equation. It allows us to ask about localization of eigenstates at particular energies. We expect that high-energy eigenstates will be non-localized, while very low-energy eigenstates will be localized, with a crossover energy called the mobility edge. This integral may not, in general, converge, so we can analytically continue it to the upper half-plane and work with $G(x, y, E + i\eta, V)$ for $\eta > 0$ in the upper half-plane. We see that only localized states contribute to

$$\lim_{\eta \to 0^+} G(x, y, E + i\eta, V)$$

since this is related to

$$\lim_{t \to \infty} G(x, y, t, V)$$

The randomness of V causes the phase of $G(x, y, E, V)$ to become uniformly distributed over $0, 2\pi$ at large spatial separation, so averaging over V won't give a useful measure of the spread of the wavefunctions. Instead, we focus on

$$\overline{|G(x, y, E, V)|^2}$$

where the average is over different realizations of the random potential.

A useful quantity to consider is the density of states. McKane and Stone show that

$$\int_y \lim_{\eta\to 0^+} \eta\overline{|G(x,y,E+i\eta,V)|^2} \propto \overline{\rho_\ell(E)}$$

the density of localized states, while

$$\lim_{\eta\to 0^+} \int_y \eta\overline{|G(x,y,E+i\eta,V)|^2} \propto \overline{\rho(E)}$$

the full density of states. We see that if and only if there are localized states at E, there exist some x, y for which $|G|^2 \sim \eta^{-1}$ as $\eta \to 0$. If there are no localized states, on the other hand, we must have only $\int |G|^2 \sim \eta^{-1}$, without any pointwise divergences in $|G|^2$.

The next step is to introduce a field theory whose four-point function is precisely

$$|G(x,y,E,V)|^2$$

To that end, we introduce fields ϕ_+ and ϕ_-, with n_+ and n_- components, respectively, with Hamiltonian densities

$$\mathcal{H}_\pm = (-\nabla^2 + V - (E \pm i\eta))$$

This gives

$$G(x,y,E+i\eta,V) = \lim_{n_\pm\to 0^+} \langle \phi_+(x)\phi_+(y)\phi_-(x)\phi_-(y)\rangle$$

where now the average is over the canonical distribution for some fixed V. We can now average over V by integrating out the dependence on V, which leaves an effective interaction between the fields proportional to $(\phi_+^2 + \phi_-^2)^2$.

It is clear the resulting effective Hamiltonian is $O(n_+ + n_-)$ invariant when $\eta = 0$. However, to keep things convergent we need to instead consider the limit $\eta \to 0^+$. This limit can produce a broken symmetry phase, since η distinguishes between ϕ_+ and ϕ_-. Indeed, McKane and Stone show that the symmetry must be broken to $O(n_+) \times O(n_-)$ whenever the density of states is not zero.

Naively, one expects this broken-symmetry phase to possess Goldstone modes. Indeed, as Goldstone's theorem suggests, we find

$$\int_y |G(x,y,E)|^2 \to \infty$$

However, in this setting, it need not be because there exists a massless excitation. If $\rho_\ell = 0$ and $|G|^2$ is finite, then large $x - y$ must be contributing significantly to the integral, so there must be a massless long-range excitation. However, if $\rho_\ll \neq 0$ and $|G|^2$ itself diverges for some $|x - y|$, then even if all excitations are short range the integral will still converge. We see that the existence of localized states offers a way out of Goldstone's theorem in this case. Note, however, that the localization is relevant only in the limit $n_\pm \to 0$.

From this model, one can then go from working in terms of the $\phi_\pm$ to composite fields $Q_{\pm\pm}$ so that

$$\overline{|G^2(x,y,E+i\eta)|^2} \propto \langle \phi_+(x)\phi_+(y)\phi_-(x)\phi_-(y)\rangle \propto \langle Q_{+-}(x)Q_{-+}(y)\rangle$$

These fields are expected to have Goldstone modes, and so for low energies we can treat amplitude fluctuations as frozen out to obtain an effective nonlinear σ model. It is then possible to use renormalization group techniques to study the behavior of this theory near the mobility edge E_c in more detail.

4 Experimental demonstration

Although Anderson localization was initially discussed in the context of conduction of electrons, electron-electron interactions make the actual situation more complex than the theory discussed above. [4] However, the same dynamics are present in more experimentally tractable systems. Early results came from photonics [7], but more recent advances in control of ultracold atoms have allowed more detailed studies of the dynamics of the localized system.

Semeghini et al [8] study a three-dimensional Bose-Einstein condensate of ultracold potassium-39. The atoms are initially trapped in a harmonic potential and cooled using Feshbach resonances controlled by a magnetic field. Interference of lasers is used to create a random "speckle potential" with correlation length $\sigma_R \sim 10^{-6}m$. The system size is on the order of $10^{-3}m$, so over system-sized scales this potential is essentially i.i.d. However, these short-range correlations do introduce an energy scale

$$E_R = \frac{\hbar^2}{m\sigma_R^2}$$

the effect of which will be discussed below. Adjusting the intensity of the lasers allows the experimenters to control the strength of the disorder.

At the start of the experiment, the interactions and the trap are turned off, leaving the atoms free to diffuse through the speckle potential. By measuring the spread of the distribution of atoms over time, they can determine the effective diffusion rate. Results are shown in figure 4. For pure diffusion, the spatial variance should increase linearly with time, while for atoms in localized states it should level off at some constant value. We see that localization occurs for large disorder, but not for small disorder. There is also an intermediate regime in which variance grows sublinearly, but apparently monotonically. This may be due to either finite system size or finite observation time.

This experimental setup also makes it possible to study the structure of the mobility edge. First, the kinetic energy distribution of the atoms can be measured, and these data can be combined with numerics to fit a density of states $n(E)$. Next, the lasers can be used to add some quantized energy to the initial Bose-Einstein Condensate. This produces a distribution $(1-p)n(E) + pn(E-\hbar\omega)$, where p is the fraction of atoms which absorb are excited and $\hbar\omega$ is

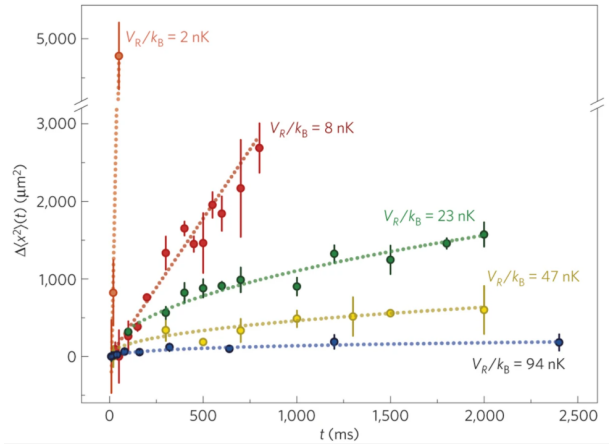

Figure 2: From [8]. Measured spatial variance over time for various disorder strengths. For weak disorder, diffusion dominates and variance grows linearly. Strong disorder causes a plateau at some finite variance corresponding to the size of the localized states.

the energy of the excitations. By varying ω, the experimenters can then adjust the initial distribution of energies.

This known initial energy distribution is then allowed to diffuse through the speckle potential. Atoms which have energies above the mobility edge are lost by the system, while those with energies below the mobility edge remain trapped. By measuring the fraction of atoms which remain, the fraction of the original distribution which was above the mobility edge is determined. Since the initial distribution is known, the location of the mobility edge can be estimated. Using several values of ω improves experimental accuracy.

Results are shown in figure 4. For weak disorder, the mobility edge E_c is quite close to the characteristic disorder strength V. However, for strong disorder, the energy scale E_R associated with the autocorrelation of the potential becomes relevant. We see that states with $E > E_R$ are mobile no matter how strong the disorder is. This is largely consistent with prior numerical work, with diferences that the authors suggest may be explained by the anisotropy of their speckle potential.

5 Addition of weak interactions

Fleishman and Anderson [4] study the effects of electron-electron interactions on a localized system. Consider a Hamiltonian with localized wavefunctions $\psi_a(x)$. Let $V(r)$ be some interaction potential and define

$$U_{abcd} = \int_{x,y} \psi_a(x)\psi_b(x)V(x-y)\psi_c(y)\psi_d(y)$$

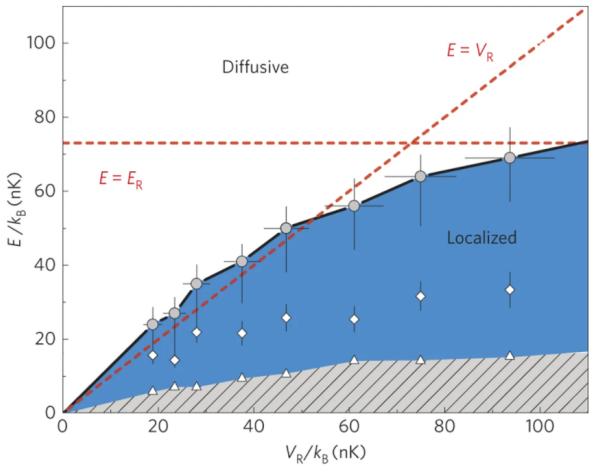

Figure 3: From [8]. Measurements of the energy of the mobility edge for various amounts of disorder. For weak disorder, we see $E_c \sim V$, but for sufficiently strong disorder the nonzero spatial autocorrelation of the speckle potential becomes important.

We then add an interaction term to the Hamiltonian

$$\sum_{a,b,c,d} U_{abcd} c_a^\dagger c_b^\dagger c_c c_d$$

Treating this interaction perturbatively, it is possible to study the Green's functions of the resulting theory.

The analysis finds that the physical consequences depend heavily on the range of the interaction. For a short-range interaction, there exist quasiparticle excitations both above and below the mobility edge. Those below are discrete bound states, while those above are a continuum of conducting states. Low-energy excitations are thus nonconducting. Furthermore, these low-energy excitations are stable in this case. If the interaction is long-range, on the other hand, the excitations form a continuum at all energies. The ground state remains nonconducting, and the low-energy excitations are not stable.

5.1 Numerical results

Delande et al [3] study the effect of interactions on a many-body Anderson-localized system. They consider a one-dimensional system of $N = 25$ bosons with an attractive two-body interaction of the form

$$\int \alpha^\dagger(x)\alpha^\dagger(x)\alpha(x)\alpha(x)$$

This interaction is of course maximally short-range. At the level of mean field theory, the ground state finds the particles all clumped into a single soliton. One

then expects that this effective single-particle system can undergo Anderson localization.[6] In [3], the authors consider an initial soliton state, then "turn on" both many-body effects and a random speckle potential. The goal is to study the influence of the many-body effects on the resulting dynamics.

The first necessary simplification is the discretization of spacetime. Although the original system of interest was continuous, computational tractability requires moving to a tight-binding lattice model. The authors choose a discretization scale one-fifth the characteristic size of the soliton, ξ. Time is also discretized. The space must be finite in extent (1921 lattice points are used). The authors also choose a speckle potential with correlation length $\frac{\xi}{4}$. They find that it is not necessary to represent the full Fock space, since the bosons are never all localized on the same site; instead, they restrict themselves to the subspace with at most 14 particles per site. Most critically, the full exponential Hilbert space is much too large to represent. Instead, Matrix Product states with bond dimension $\chi = 30$ are used. This algorithm is considered "quasi-exact": In the limit of small space and time scales, large space and time domains, large bond dimension, and large Fock spaces, the results should converge to the exact physical solution. The authors mention convergence studies on many of the parameters above to ensure that errors are reasonably small.

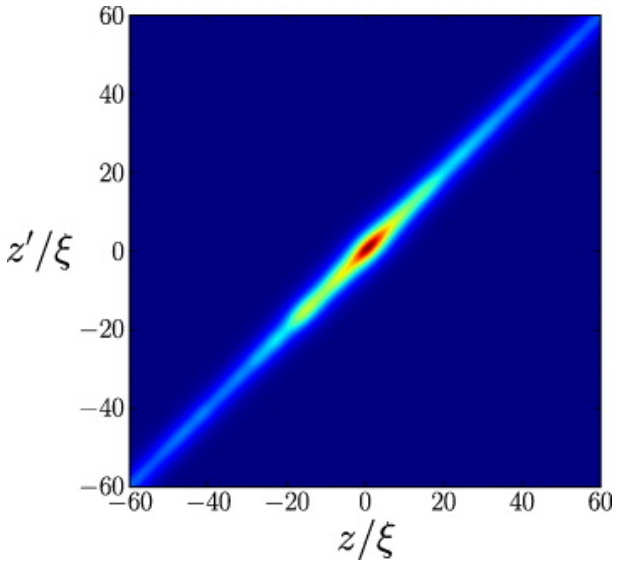

Figure 4: From [3]. Numerical estimates of the effective one-body density matrix after long times. We see that the density matrix is almost diagonal, with concentration near $0, 0$. This indicates that the soliton survives after many-body interactions and the disordered potential are turned on.

The key finding (Fig 5.1) is that the particles remain strongly localized near their initial configuration even for long times. Furthermore, Fig 5.1 shows that the localization in the full many-body case is essentially indistinguishable from

the results of the mean-field treatment. However, analysis of the one-body density matrix finds that its largest eigenvalue after long time is 0.14. In other words, there are many-body effects present which create important differences from the mean-field picture, but these effects don't change localization.

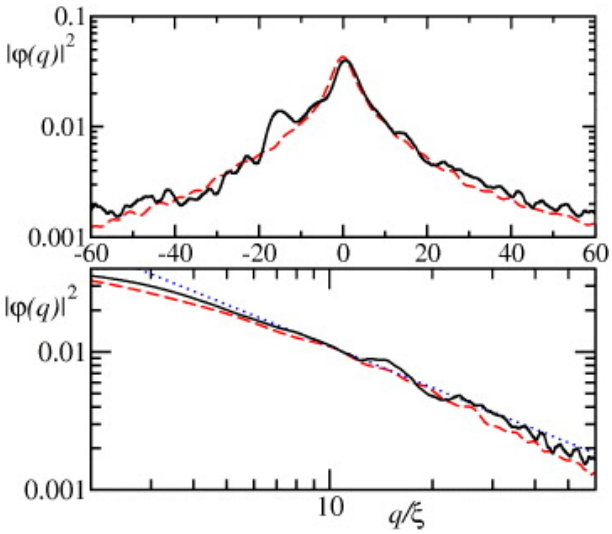

Figure 5: From [3]. Probability density function for the location of the center of mass after long time. In blue, computed in the full many-body theory; in red, computed in the effective one-body soliton theory. The many-body effects seem to have no influence on how localized the system is.

6 Conclusions

Disorder in physical systems has interesting and surprising properties. In particular, random fluctuations to a potential can break ergodicity and prevent transport by creating localized states. In three dimensions, this occurs only when the fluctuations are strong compared to the strength and range of the hopping terms. By integrating out the random potential, an interacting effective field theory describing the diffusion properties is obtained. This field theory has unusual and interesting behaviors when there are localized states in the underlying system. Experimental tests of Anderson localization show qualitative agreement with theoretical predictions, and agreement seems to be limited mostly by the imprecise nature of the theory. When electron-electron interactions are added, the analyses gets quite tricky, but one can show that the resulting quasiparticle excitations don't dramatically change the physical properties of the system, at least for low energies and weak interactions. Numerical studies of short-range attractive boson-boson interactions find that localization

still occurs in that case as well, with good agreement between mean field theory and a full many-body treatment.

References

[1] Dmitry A. Abanin and Zlatko Papić. "Recent progress in many-body localization". In: *Annalen der Physik* 529.7 (2017), p. 1700169. DOI: 10.1002/andp.201700169.

[2] P. W. Anderson. "Absence of Diffusion in Certain Random Lattices". In: *Physical Review* 109.5 (1958), pp. 1492–1505. DOI: 10.1103/physrev.109.1492. URL: https://journals.aps.org/pr/abstract/10.1103/PhysRev.109.1492.

[3] Dominique Delande et al. "Many-body Anderson localization in one-dimensional systems". In: *New Journal of Physics* 15.4 (2013), p. 045021. DOI: 10.1088/1367-2630/15/4/045021. URL: https://iopscience.iop.org/article/10.1088/1367-2630/15/4/045021.

[4] L. Fleishman and P. W. Anderson. "Interactions and the Anderson transition". In: *Physical Review B* 21.6 (1980), pp. 2366–2377. DOI: 10.1103/physrevb.21.2366. URL: https://journals.aps.org/prb/abstract/10.1103/PhysRevB.21.2366.

[5] Benjamin Sacépé et al. "Localization of preformed Cooper pairs in disordered superconductors". In: *Nature Physics* 7.3 (2011), pp. 239–244. DOI: 10.1038/nphys1892.

[6] Krzysztof Sacha et al. "Anderson Localization of Solitons". In: *Physical Review Letters* 103.21 (2009). DOI: 10.1103/physrevlett.103.210402. URL: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.103.210402.

[7] Tal Schwartz et al. "Transport and Anderson localization in disordered two-dimensional photonic lattices". In: *Nature* 446.7131 (2007), pp. 52–55. DOI: 10.1038/nature05623.

[8] G. Semeghini et al. "Measurement of the mobility edge for 3D Anderson localization". In: *Nature Physics* 11.7 (2015), pp. 554–559. DOI: 10.1038/nphys3339. URL: https://www.nature.com/articles/nphys3339.