# Cooper pairs in Superfluid $^3He$

Penghao Zhu

May 10, 2018

**Abstract**

In this paper, I will talk about why the Cooper pairs in superfluid $^3He$ is spin triplet pairing and P-wave. I mainly argue from the aspect that how the theory match with the experiment data on spin susceptibility.

# 0 Introduction

Helium has two (stable) isotope $^3He$ and $^4He$. The superfluid of $^4He$ was discovered in 1938 and was well understood long before it was shown in lab. However, superfluid of $^3He$ is much more complex. Its liquid form was obtained around 1950. People started to generalize the BCS theory to liquid $^3He$ to understand its superfluid phase in late fifties. As we now know the Cooper pair in superfluid phase of $^3He$ is P-wave and spin triplet coupling. However, it is a long journey to reach this conclusion theoretically and the work behind it is considerable. In this paper, I want to talk about how to understand the superfluid phase of of $^3He$ is P-wave and spin triplet pairing in a loose but heuristic way.

# 1 Important experimental facts

Figure 1 shows the phase digram of Helium-3 in P-T plane with zero external field [1]. At low pressure and between about 100mk and 3 mk, liquid $^3He$ is described by Landau's theory of normal fermi liquid(N)[2]. There are two types of superfluid phase: $^3He - A$ and $^3He - B$. The N-A and N-B transition is a second order transition while the A-B transition is a first order transition. We can see from Figure 1 that **the N-B transition curve and N-A transition curve in P-T plane seems to be the same curve (i.e. they are continuously connected).**

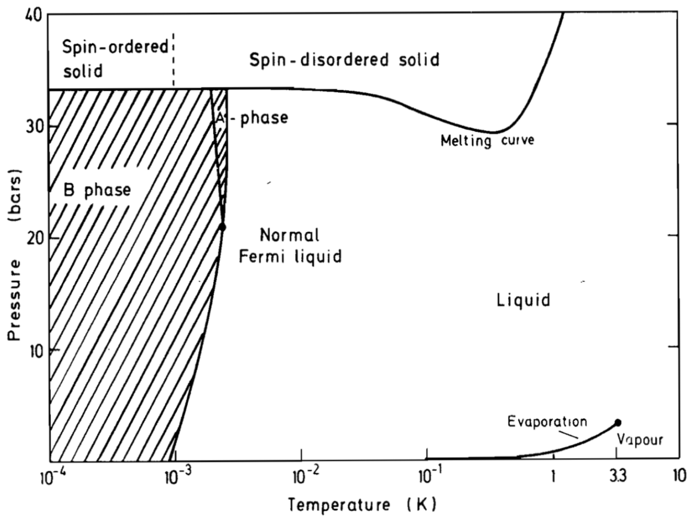


Figure 1: Pressure versus temperature phase diagram for Helium-3

Figure 2 shows the spin susceptibility of $^3He-B$[1], in which we can see $\chi_B(T)$ **decreases with temperature, and as $T \to 0$, $\chi_B \sim \chi_N/3 \neq 0$.** And the **spin suscepbtility of $^3He-A$ is independent of temperature and close to that of normal fermi liquid.**

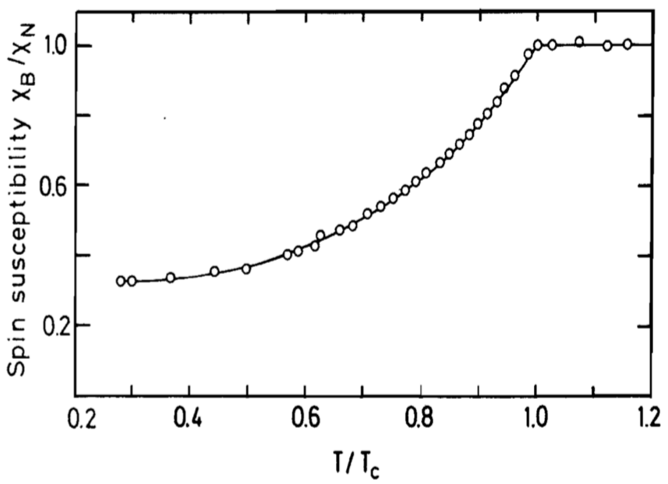


Figure 2: Magnetic susceptibility of 'He-B as a function of temperature at 20bar

# 2 The Cooper instability

Let us ask an interesting question(Cooper problem [3]): if we let two fermions near the fermi surface interact with each other, will the lowest energy of this two particle states less than $2\epsilon_F$?( where $\epsilon_F$ is the fermi energy). Begin from Schrodinger equation for a 2-body system:

$$\left[-\frac{\hbar^2}{m} + V(\mathbf{r})\right]\phi(\mathbf{r}) = E'\phi(\mathbf{r}), \tag{1}$$

do the Fourier transformation

$$\phi(\mathbf{r}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}), \quad V_{\mathbf{k}\mathbf{k}'} = \int d\mathbf{r} \exp\left[-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right] V(\mathbf{r}), \tag{2}$$

and set $\epsilon_k = (\hbar^2/2m)(k^2 - k_F^2)$ and $E = E' - \hbar^2 k_F^2/2m$. Since $V_{\mathbf{k}\mathbf{k}'}$ only depends on $|\mathbf{k}-\mathbf{k}'|^2 = k^2 + k'^2 - 2kk'\cos\theta$, we can expand it as

$$V_{\mathbf{k}\mathbf{k}'} = \sum_l (2l+1) V_l(k,k') P_l(\cos\theta). \tag{3}$$

Simultaneously, we can expand $\phi_{\mathbf{k}}$ as $\phi_{\mathbf{k}} = \sum_l \psi_l(\mathbf{k}) Y_{lm}(\hat{\mathbf{k}})$. Plug all these in the Fourier transformed Schrodinger equation, we can get

$$(2\epsilon_{\mathbf{k}} - E)\psi_l(\mathbf{k}) = -\frac{1}{(2\pi)^3}\int_{k_F}^{\infty} 4\pi k'^2 V_l(k,k')\psi_l(k')dk'. \tag{4}$$

Note that $k$ and $k'$ are both $\gtrsim k_F$. Whether eqn(4) has a solution with negative $E$ (i.e pairing state have energy less than $2\epsilon_F$) depends on the details of $V_l(k,k')$. Let us use the approximation used in [4]:

$$\begin{aligned} V_l(k,k') &= V_l \quad k_F - \Delta k \leqslant k, k' \leqslant k_F + \Delta k \\ &= 0 \quad otherwise \end{aligned} \tag{5}$$

then we will get

$$(2\epsilon_{\mathbf{k}} - E)\psi_l(\mathbf{k}) = -\frac{1}{2}V_l(dn/d\epsilon)\int_0^{(\hbar/m)k_F\Delta k} \psi_l(k')d\epsilon_{k'}. \tag{6}$$

From above equation we can see for $V_l < 0$ and $|V_l|$ large enough, $E < 0$ which means the pairing state will be energetically advantageous. Thus the filled fermi sea is unstable, that is why we call this **Cooper instability**. By solving eqn(6), we can get

$$1 = -\frac{1}{2}V_l(dn/d\epsilon)\int_0^{\epsilon_c} \frac{N(\epsilon_{k'})d\epsilon_{k'}}{2\epsilon_{k'} - E}, \tag{7}$$

where $\epsilon_c = (\hbar/m)k_F\Delta k$. At finite temperature, fermions satisfy the fermi statistics:

$$n_k = \frac{1}{e^{\beta E_k} + 1}, \tag{8}$$

then $N(\epsilon_k)$ which is the probability of forming a pairing state could be written as $(1-n_k)^2$. Now we want to calculate transition temperature at which E begin to become negative. Plug $E = 0$ into eqn(7), we can then get the transition temperature is

$$k_B T_0 \sim \epsilon_c exp(2/(V_l dn/d\epsilon). \tag{9}$$

From the above equation we can see that the smallest $V_l$ for different $l$ will dominate the transition, and we usually say that the system has instability $l$. $l$ is angular momentum of cooper pairs (e.g. $l = 0$ or s-wave in BCS superconductor). And intuitively,

$$V_l \sim \int V(\mathbf{r}) R_l^2(k_F r) r^2 dr, \tag{10}$$

where $R_l$ is the radial part of $l$-orbital wave function, so $R_l^2(k_F r)r^2$ gives the density of fermions. From eqn(10) and eqn(9), we can get some intuitive understanding about cooper pairs in superfluid $^3He$: because " the interaction potential between two He atoms is strong repulsive at short distances and becomes attractive only for inter-atom separation $r \sim r_0 \sim 3\mathring{A}$ "[5], the $V(\mathbf{r})$ should be positive (repulsive) when $\mathbf{r}$ is small. Given that for $l = 0$, $R_0(k_F r_0) \sim 0$, the $V_0$ should be positive and then the instability could not be $l = 0$. **We can then draw a conclusion that the cooper pairs in superfluid $^3He$ cannot be P-wave.**

# 3 BCS theory at finite temperature and its generalization to anisotropic superfluid

## 3.1 Original BCS theory: spin singlet pairing

I think the central part of BCS theory is to write down appropriate wave-function to describe pairing states. In the original BCS theory, states $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ form the cooper pair. We now focus on 3 states: "ground pair"(GP), "broken pair" (BP) and "excited pair"(EP). We have seen that GP state has lower energy then GP, so we could understand BP and EP are two different excited states. Bardeen *et al.*[4] write the wave-function as

$$\Psi = \prod_{\mathbf{k}} \Phi_{\mathbf{k}} \tag{11}$$

where

$$\begin{aligned} \Phi_{\mathbf{k},GP} &= u_{\mathbf{k}}(T)|0,0\rangle_{\mathbf{k}} + v_{\mathbf{k}}(T)|1,1\rangle_{\mathbf{k}} \\ \Phi_{\mathbf{k},EP} &= u_{\mathbf{k}}(T)|0,0\rangle_{\mathbf{k}} - v_{\mathbf{k}}(T)|1,1\rangle_{\mathbf{k}} \\ |u_{\mathbf{k}}|^2 &+ |v_{\mathbf{k}}|^2 = 1 \end{aligned} \tag{12}$$

and define

$$\begin{aligned} v_{\mathbf{k}}(T) &= \Delta_{\mathbf{k}} / \left[ |\Delta_{\mathbf{k}}|^2 + (E_{\mathbf{k}} + \epsilon_{\mathbf{k}})^2 \right]^{1/2} \\ u_{\mathbf{k}}(T) &= (E_{\mathbf{k}} + \epsilon_{\mathbf{k}}) / \left[ |\Delta_{\mathbf{k}}|^2 + (E_{\mathbf{k}} + \epsilon_{\mathbf{k}})^2 \right]^{1/2} \\ E_{\mathbf{k}}(T) &= \left[ \epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}(T)|^2 \right]^{1/2} \end{aligned} \tag{13}$$

where $\Delta_{\mathbf{k}}$ satisfy the gap equation. Note that $|0,1\rangle_{\mathbf{k}}$ means $\mathbf{k}\uparrow$ is occupied and $-\mathbf{k}\downarrow$ is empty. After some math, we can get

$$E_{EP} - E_{GP} = 2E_k, \quad E_{BP} - E_{GP} = E_k. \tag{14}$$

Then we can write down the partition function(without external field) of this system

$$Z = 1 + 2e^{-\beta E_k(T)} + e^{-2\beta E_k(T)}, \tag{15}$$

it is easy to see if we add an external magnetic field $H$, then the partition function will be

$$Z = 1 + e^{-\beta(E_k(T) - \frac{1}{2}\mu H)} + e^{-\beta(E_k(T) + \frac{1}{2}\mu H)} + e^{-2\beta E_k(T)}. \tag{16}$$

where $\mu = ge\hbar/2m$ is the magneton. With partition function, we can then calculate thermodynamic quantities such as spin susceptibility:

$$\chi(T) = \frac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)\left[ Y(T) / \left( 1 + \frac{1}{4}Z_0 Y(T) \right) \right], \tag{17}$$

where $\gamma = 2\mu/\hbar$ and $Y(T)$ is known as Yosida function. Its temperature dependence is shown in figure 3. It is helpful to make some comments here:

1. Remember that eqn(17) is for spin singlet pairing.

2. When $T \to T_c$, $\chi(T) \to \frac{\chi_0}{1+[\gamma^{-2}\hbar{-2}(dn/d\epsilon)^{-1}Z_0\chi_0]}$, which is what we get in normal fermi liquid case; when $T \to 0$, $\chi(T) \to \chi_0$. $\chi_0 = \frac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)Y(T)$ is the spin susceptibility of free fermion gas, and $Z_0$ here is the magnitude of coupling between spins (molecular field [2]).

3. From figure 3, we can see that when $T \to 0$,$\chi(0) = 0$.

4. $\frac{d}{dT}\chi(T) = \frac{Y'}{\left(1-\frac{3}{4}Y\right)^2} > 0$ because $Y' > 0$, *i,e,* the spin susceptibility of singlet paring will decrease with temperature.

Comparing comment 3, 4 with experimental results referred in Section 1: a) $\chi_A$ is T-independent and close to $\chi_N$; b) $\chi_B(T)$ decreases with temperature but have $\chi_B(0) \sim \chi_N/3 \neq 0$. We can draw a conclusion that **both** $^3He - A$ **and** $^3He - B$ **cannot be singlet pairing, i.e. they must be triplet pairing.** Thus, $l$ **must be odd** because of the antisymmetric property of the wave-fucntion.

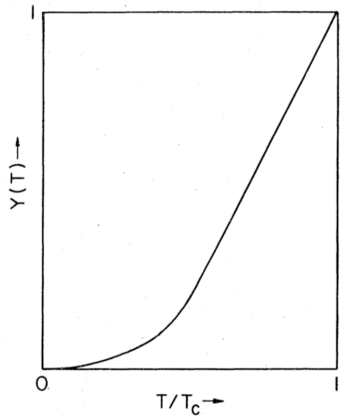


Figure 3: Temperature dependence of Yosida function Y(T)

## 3.2 Spin triplet pairing

Most generally, the wave-function of pairing state could be written as

$$\Psi_{pair} = F_{\uparrow\uparrow}(\mathbf{r})|\uparrow\uparrow\rangle + F_{\downarrow\downarrow}(\mathbf{r})|\downarrow\downarrow\rangle + F_{\downarrow\uparrow}(\mathbf{r})\frac{1}{\sqrt{2}}(|\uparrow\downarrow + |\downarrow\uparrow\rangle), \tag{18}$$

where $\mathbf{r}$ is the relative coordinates. If we choose a special case that only fermions with same spin projection can pair with each other, then $F_{\uparrow\downarrow} = 0$. This state is usually called "equal-spin-pairing"(ESP). We can now do similar thing as we do in singlet pairing to calculate the spin susceptibility of ESP state, because now the spin part of wave-function is not trivial comparing with that in singlet pairing, generally the state of the system could be written as

$$\hat{\Psi}(\mathbf{n}) = \Psi \hat{f}(\mathbf{n}), \tag{19}$$

where $\mathbf{n}$ is the spin axes. Or equivalently,

$$\hat{\Psi}(\mathbf{n}) = \Psi i \sum_{j=1}^{3} (\sigma_i \sigma_2)_{\alpha\beta} d_i(\mathbf{n}), \tag{20}$$

where $\mathbf{d}(\mathbf{n})$ is normalized. Because the math is complex, I will not go into it and directly give the result: $\chi_{ESP} = \chi_N$. Another special choice is BW state (named after Balian and Werthamer), which is isotropic. The idea is that in $l = 1$, we can choose $F_{\uparrow\uparrow}(\mathbf{r})$ to correspond to angular momentum $L_z = -1$, and $F_{\downarrow\downarrow}(\mathbf{r}), F_{\downarrow\uparrow}(\mathbf{r})$ correspond to $L_z = -1, 0$ respectively [5]. Then we will have $J = 0$ which means all its properties should be isotropic. And the spin susceptibility for this state is[2]

$$\chi_{BW}/\chi_N = (1 + \frac{1}{4}Z_0)\left[\frac{2}{3} + \frac{Y(T)}{3}\right] / \left\{1 + \frac{1}{4}Z_0\left[\frac{2}{3} + \frac{Y(T)}{3}\right]\right\}. \tag{21}$$

As we can see, $\chi_{BW}$ decreases with temperature and when $T \to 0$, $\chi_{BW}/\chi_N \to (2/3 + Z_0/6)/(1 + Z_0/6)$. Given that $Z_0 \sim 3$, $\chi_{BW}/\chi_N \to 1/3$ at $T = 0$ ! Then from the aspect of spin susceptibility, we can loosely judge that $^3He - B$ **corresponds to BW state (so $l = 1$) and $^3He - A$ corresponds to ESP state.**

Up to now, we have had several conclusions:

1. $l \neq 0$;
2. $^3He - A$ and $^3He - B$ phase must correspond to triplet pairing and then $l$ must be odd.
3. From the point view of spin susceptibility, $^3He - B$ corresponds to BW state and $^3He - A$ correspond to ESP state.

The logic link between 2 and 3 is that 2 gives some constraints on the possibilities of $^3He - A$ and $^3He - B$. Then we can find two special state ESP and BW satisfying the constraints corresponds to $^3He - A$ and $^3He - B$ respectively, and it is consistent with experimental results of spin susceptibility. Note that the isotropic properties of spin susceptibility of $^3He - B$ phase plus conclusion 2 require that $l = 1$ for $^3He - B$ phase. Remember that in figure 1, we see that the A-N transition curve and B-N transition curve is continuously connected which is actually one curve. This suggests that A-N transition and B-N transition should correspond to the same Cooper instability in normal fermi liquid, which means they have the same dominant $V_l$ and so same $l$. From this argument, we can say $l$ is also equal to 1 for $^3He-A$ phase. Let us now proceed with these conclusions to see what problems we will meet: as we know, when $T \lesssim T_c$, the system could be described by Ginzburg-Landau theory with free energy of the form $\alpha(t)\Psi^2+\beta\Psi^4$, where $t = T-T_c/T_c$. By carefully calculate the free energy, we can find that for $l = 1$, the BW phase has lower energy than ESP phase, which means the BW state is more stable than ESP state. This conclusion could be generalized to any temperature below $T_c$[6]. If we think $^3He-B$ corresponds to BW state and $^3He-A$ corresponds to ESP state and both A-N, B-N transition are because of the same instability, then it is unreasonable for A-phase to appear, because B-phase is more stable. To solve this problem, Anderson-Brinkman provided a clever idea called spin fluctuations[7].

# 4 Spin fluctuations

The basic idea of Anderson-Brinkman is that we can add an interaction term of quasi-particle which will depend on the details of superfluid state formed, then if some specific state which is not stable originally has an attractive contribution form the newly added interaction term, it is possible that this state will become stable. Now the question becomes: what interaction that physically makes sense can we add? Let us consider such an effect: a $^3He$ atom at point $\mathbf{r}$ and time t with spin $\mathbf{S}(\mathbf{r},t)$ will generate a molecular field

$$\mathbf{H}_{mol}(\mathbf{r},t) = -\gamma^{-1}\hbar^{-2}(dn/d\epsilon)^{-1}Z_0\mathbf{S}(\mathbf{r},t) \tag{22}$$

the molecular field comes from the expansion of 2-body interaction of quasi-particles in Landau's fermi liquid theory, $Z_0$ is the Landau parameter. Then this molecular field will produces a spin polarization of the neighboring liquid. This process repeat again and again so that it can be described by a spin-dependent effective interaction between two particles. For triplet pairing, generally the spin susceptibility is anisotropic which should be described by a tensor. Give $\mathbf{H}_{mol}(\mathbf{r},t)$ and $\mathbf{S}(\mathbf{r},t) = \hbar\boldsymbol{\delta}$, at point $\mathbf{r}'$ and time $t'$

$$\begin{aligned} M_i(\mathbf{r}',t') &= \gamma^2\hbar^2 \int \chi_{ij}(\mathbf{r}'-\mathbf{r},t'-t)H_{mol,j}(\mathbf{r},t) \\ &= -\chi_{ij}(\mathbf{r}'-\mathbf{r},t'-t)\gamma\hbar\zeta_0\boldsymbol{\delta}_j, \end{aligned} \tag{23}$$

where $\zeta_0 = (dn/d\epsilon)^{-1}Z_0$. Then the molecular field at $\mathbf{r}', t'$ should be

$$H_{mol,i}(\mathbf{r}',t') = -\chi_{ij}(\mathbf{r}'-\mathbf{r},t'-t)(\gamma\hbar)^{-1}\zeta_0^2\boldsymbol{\delta}_j, \tag{24}$$

then the change of the total energy is

$$\Delta E = -\gamma\hbar\boldsymbol{\delta}'_i H_{mol,i}(\mathbf{r}',t') = -\zeta_0^2\boldsymbol{\delta}'_i\boldsymbol{\delta}_j\chi_{ij}(\mathbf{r}'-\mathbf{r},t'-t) \tag{25}$$

Note that $\chi_{ij}(\mathbf{r}' - \mathbf{r}, t' - t) \sim \theta(t' - t)$, so when writing the effective interaction, we need to replace $\chi_{ij}(\mathbf{r}' - \mathbf{r}$ by $1/2(\chi_{ij}(\mathbf{r}' - \mathbf{r}, t' - t) + \chi_{ji}(\mathbf{r} - \mathbf{r}', t - t'))$. Thus,

$$V_{eff}(\mathbf{r}' - \mathbf{r}, t' - t) \sim \theta(t' - t) = -\zeta_0^2 \boldsymbol{\delta}_i' \boldsymbol{\delta}_j \frac{1}{2}(\chi_{ij}(\mathbf{r}' - \mathbf{r}, t' - t) + \chi_{ji}(\mathbf{r} - \mathbf{r}', t - t'). \quad (26)$$

Then the change of free energy could be calculated by $\Delta F = \langle V_{eff} \rangle$. Clearly, this will depend on the details of the spin part of the wave-fucntion, i,e, $f(\mathbf{n})$ or $\mathbf{d}(\mathbf{n})$. From calculation, we can find that for one special and highly anisotropic ESP state with $F_{\uparrow\uparrow} = e^{i\phi} F_{\downarrow\downarrow}$, the correction to free energy because of spin fluctuation feedback will make it a stable state. Then the problem encountered in the end of last section is solved! This state is know as ABM state which is named after Anderson-Brinkman-Morel. Thus, by introducing interactions described by eqn(26), the stable state (without external field) of superfluid with P-wave ($l = 1$) pairing will be either BW state or the ABM state. Together with discussion in previous sections, we can give the conclusion that $^3He - A$ **and $^3He - B$ correspond to ABM and BW states respectively.**

Finally, I want to briefly talk about how firm is this widely believed conclusion. Firstly, for $^3He - A$, the hypothesis that it is ABM phase is highly compatible with NMR(Nuclear Magnetic Resonance ) data. But the data on density of superfluid agree roughly with the theory, it is believed that this is due to the effect of geometry which is not totally understood. In this paper, I did not talk about the phase transition when exist external fields. The theory and experiment match very well on the A transition in an external field. Next, for $^3He - B$, most of the experiment data is compatible with what gives by BW phase. However, the main problem is that only near melting curve, the data of static susceptibility fit the hypothesis well, in lower pressure, there is discrepancy between theory and experiment.

# 5 conclusion

In this paper, I mainly use the experimental result for spin susceptibility to argue that the superfluid phase of $^3He$ is spin triplet pairing. For why it should be $l = 1$, I only give some naive and loose argument. By shallowly involving Cooper instability, anisotropic BCS theory and the Spin fluctuations, this paper construct a simple and heuristic way to understand the properties of Cooper pair in superfluid $^3He$.

# Reference


[1] D. Vollhardt and P. Wölfle, *The superfluid phases of helium 3*. Courier Corporation, 2013.

[2] A. J. Leggett, “A theoretical description of the new phases of liquid he 3,” *Reviews of Modern Physics*, vol. 47, no. 2, p. 331, 1975.

[3] L. N. Cooper, “Bound electron pairs in a degenerate fermi gas,” *Physical Review*, vol. 104, no. 4, p. 1189, 1956.

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, “Theory of superconductivity,” *Physical Review*, vol. 108, no. 5, p. 1175, 1957.

[5] A. J. Leggett, “Nobel lecture: Superfluid he 3: the early days as seen by a theorist,” *Reviews of modern physics*, vol. 76, no. 3, p. 999, 2004.

[6] R. Balian and N. Werthamer, “Superconductivity with pairs in a relative p wave,” *Physical review*, vol. 131, no. 4, p. 1553, 1963.

[7] P. W. Anderson and W. Brinkman, “Anisotropic superfluidity in he 3: A possible interpretation of its stability as a spin-fluctuation effect,” *Physical Review Letters*, vol. 30, no. 22, p. 1108, 1973.