# Fractional Excitations and the Quantum Hall Effect

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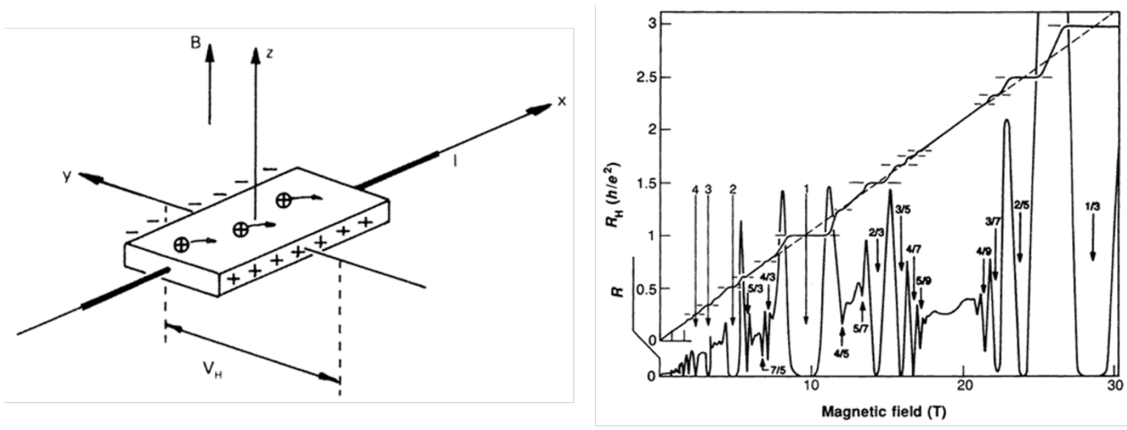
## Abstract

The fractional quantum Hall effect explains the fractional quantization of transverse conductance in the presence of a magnetic field. This effect has become of interest due to the presence of fractional excitations and other exotic emergent phenomena. In this essay, we introduce the motivation for the variational wavefunction first proposed by Laughlin, which predicts the existence of quasi-particles with fractional charge and fractional statistics, and also explains several values of the Hall quantization. We finally construct a top-down Chern-Simons theory for the FQHE in the spirit of a Landau-Ginzburg functional, and reproduce some of the phenomenology.




(a) (b)

Figure 1: a) Schematic quantum Hall effect setup, which results in a conductance transverse to the direction of the field. b) Summary of several experimental signatures of the quantum Hall effect. The quantized filling fraction, $\nu$ can be seen from the plateaus in the resistance curve

# Introduction

The class of Hall effects is an emergent electronic phenomena where two-dimensional materials obtain a transverse conductance in the presence of an external magnetic field. The effect is named after Edwin Hall, who first discovered the phenomena in 1897 in gold plates. At the time it was believed that the transverse conductance can be any possible value, but in 1975, it was shown using quantum mechanical transport calculations that the Hall conductance must be quantized in terms of the fundamental constants

$$\sigma_{xy} = \frac{e^2}{h}\nu \tag{1}$$

where we will see later that $\nu$ is the filling fraction of a quantum Hall states. In 1980, this was confirmed experimentally by v. Klitzing et. al when they were studying silicon-metal-oxide semiconductor systems. In 1983 experiments on *GaAs-GaAlAs* heterostructures showed that the Hall conductance is quantized in terms of rational numbers (rather than integers), which eventually gave rise to the effect being known as the fractional quantum Hall effect (FQHE) [7].

Conventionally, the integer quantum Hall effect (IQHE) could be understood with non-interacting electrons, and the FQHE could be understood by considering Coulombic interactions between electrons. While this has made the fractional quantum Hall problem much more difficult to solve exactly, it has introduced the system to many exotic collective phenomena not present in many fundamental theories, such as fractionally charged quasi-particles, quasi-particles with fractional exchange statistics, composite bosons, etc. Some of these phe-

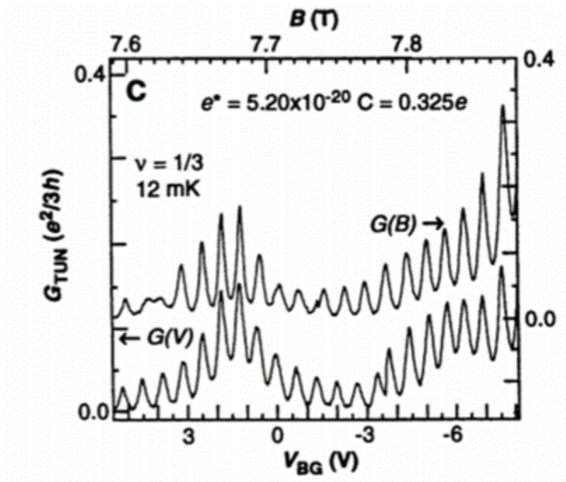



Figure 2: Resonance tunneling data on $GaAs$ heterostructures in the $\nu = \frac{1}{3}$ state. The charge measured is close to the charge of $q = \frac{e}{3}$ predicted in the section discussing the Laughlin wavefunction [2].

nomena have been tested experimentally. Both fractional and integer QHE also have strong connections with topology and the phenomonena of topological insulators/superconductors. The phenomenology may even have practical applications as proposals have been made about using these emergent properties to develop quantum computers [6].

This essay will mainly be discussing the theory of emergent behavior of fractional excitations in the FQHE, using the IQHE to motivate the discussion.

# Landau Levels

To understand the motivation for the FQHE, one needs to understand the mathematical form of Landau levels in the IQHE. As mentioned earlier, the IQHE can be understood by considering non-interacting fermions in the presence of an electromagnetic field. A Hamiltonian for a material in the $x-y$ plane with a magnetic field pointing in the $e_z$ direction is

$$H = \frac{\pi_x^2 + \pi_y^2}{2m} \tag{2}$$

$$= \frac{(p_x - \frac{eB}{2}y)^2}{2m} + \frac{(p_y + \frac{eB}{2}x)^2}{2m} \tag{3}$$

where $\boldsymbol{\pi}$ is the canonical momentum, $\mathbf{p}$ is the momentum, $e$ is the electric charge, $B$ is the external magnetic field, and $m$ is the mass of the charge carriers [8]. Note that have used units such that the speed of light, $c$ is unity. The previous Hamiltonian is merely a free-particle

Hamiltonian with the gauge choice $\mathbf{A} = -\frac{B}{2} y e_x + \frac{B}{2} x e_y$, where $\mathbf{A}$ is the vector potential. The Hamiltonian is like that of a quantum harmonic oscillator with $\pi$ commutators of

$$[\pi_x, \pi_y] = -ie\hbar B \tag{4}$$

Using these commutation relations, one can construct a pair of creation/annihilation (which are Hermitian conjugates of each other) operators which are of the form

$$\begin{aligned} a &= \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) && (5)\\ &= \frac{1}{\sqrt{2e\hbar B}}\left[-i\hbar\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) - \frac{eB}{2}(y + ix)\right] && (6)\\ &= -i\sqrt{2}\left(\ell_B \partial_{\bar{z}} + \frac{z}{4\ell_B}\right) && (7) \end{aligned}$$

where in the last line, we introduced the *magnetic length*. $\ell_B^2 \equiv \frac{\hbar}{eB}$ and the holomorphic coordinates of

$$z = x - iy \qquad \bar{z} = x + iy \qquad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{8}$$

Plugging these creation/annihilation operators back into the Hamiltonian in Eq. (2) and solving the time independent Schrödinger equation yields a ground state much like that of quantum harmonic oscillator [8].

$$\psi_{\mathrm{LLL}}(z, \bar{z}) = f(z) e^{-|z|^2/4\ell_B^2} \tag{9}$$

The set of solutions to the Schrödinger equation were solved by Landau and thus are known to be *Landau levels*, with the LLL subscript denoting the lowest Landau level. However, this does not take account of the degeneracy or filling of each Landau level. To do that we need to introduce another set of "non-canonical momenta" $\tilde{\boldsymbol{\pi}} \equiv \mathbf{p} - e\mathbf{A}$ [8]. The commutation relations for these non-canonical momenta combined with the canonical ones using the gauge choice above is

$$[\tilde{\pi}_x, \tilde{\pi}_y] = ie\hbar B \qquad\qquad [\pi_i, \tilde{\pi}_j] = 0 \tag{10}$$

The non-canonical momenta can also be written in terms of creation/annihilation operators just like that of Eq. (5-7).

$$\begin{aligned} b &= \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) && (11)\\ &= -i\sqrt{2}\left(\ell_B \partial_z + \frac{\bar{z}}{4\ell_B}\right) && (12) \end{aligned}$$

One thing to notice is that acting $b$ on $\psi_{\mathrm{LLL}}$ in Eq. (9) gives out a factor of $\frac{z}{2\ell_B}$ which means that one can construct the generalized degenerate (non-normalized) LLL as

$$\psi_{\mathrm{LLL}}(z, \bar{z}, m) \sim z^m e^{-|z|^2/4\ell_B^2} \tag{13}$$

If we used the polar imaginary representation of $z = re^{i\theta}$ the above wavefunction would become

$$\psi_{\mathrm{LLL}}(r, \theta, m) \sim r^m e^{im\theta} e^{-r^2/4\ell_B^2} \tag{14}$$

and so we can see that the wavefunction is peaked at a value of circle with radius $r^2 = 2m\ell_B^2$. The number of states, $\mathcal{N}$ in an area $A$ of a full Landau level is merely,

$$\mathcal{N} = \frac{AB}{\Phi_0} \tag{15}$$

with $\Phi_0$ being the flux quanta as defined in Appendix A.

# Fractional Excitations from a Trial Wavefunction

While the physics of IQHE and FQHE at first look fundamentally different, one can gain insight by looking at non-interacting Landau level physics. Laughlin's well-known trial wavefunction took the mathematical form of the LLL and extrapolated it to a many-body interacting problem. Some qualitative aspects of this trial is that the wavefunction should be odd under electron exchange due to fermionic statistics, and thus should vanish when two electrons are in the same positions. The wavefunction should also vanish the electrons are far from the origin due to the nature of the Coulombic interaction. The simplest $N$-body wavefunction that satisfies these ideas and is most similar to that of the LLL wavefunction is of the form [4]

$$\psi(z_i) = \prod_{i<j} (z_i - z_j)^m e^{-\sum_{k=1}^{N} |z_k|^2/4\ell_B^2} \tag{16}$$

One can find the number of states in this wavefunction using the same method as that of the LLL. However, due to the product of terms, this wavefunction is peaked at a radius of $r^2 = 2mN\ell_B^2$ in the thermodynamic limit. Taking this area and dividing by that of a full Landau level in Eq. (15) gives us [8]

$$\mathcal{N} = \frac{2\pi mN\ell_B^2 B}{\Phi_0} = mN \tag{17}$$

Note that $\frac{N}{\mathcal{N}}$ is the filling fraction so we find the relationship between the filling factor and the degeneracy to be $\nu = m^{-1}$. One thing to note is that the Laughlin wavefunction only works when $m$ is an odd integer, which described the only experimentally discovered $\nu = \frac{1}{3}$

state at the time quite well. Numerical diagonlization of a 3-body Hamiltonian shows a $> 99\%$ overlap with the Laughlin wavefunction for $3 \leq m \leq 11$ [4].

One of the striking predictions of the Lauglin wavefunction is the emergence of fractional excitations. One of the simplest excitations to recognize is fractional charge, which one can see by probing the quasi-hole excitations. It turns out that using similar qualitative ideas that yielded the Laughlin wavefunction, one can guess the functional form of the quasi-particle/quasi-hole excitations. Quasi-holes can be thought of as particles at positions (with an exponential spread) where electrons cannot exist, which means that the simplest wavefunction similar to that of the Laughlin wavefunction is

$$\psi^{+} = \prod_{a=1}^{N} \prod_{b=1}^{M} (z_a - \zeta_b) \prod_{i<j} (z_i - z_j)^m e^{-\sum_{k=1}^{N} |z_k|^2/4\ell_B^2} \tag{18}$$

for $M$ quasi-holes located at positions $\zeta_b$. If we took $M = m$ then the above wavefunction looks like a system of $N - 1$ electrons, with holes having an average charge of $+\frac{e}{m}$. Likewise the quasi-particles have an average charge of $-\frac{e}{m}$ [4]. Thus we have shown that fractional charges emerge in quantum Hall systems with electron interactions. One natural question to ask after obtaining these quasi-particle/quasi-holes is what are the exchange statistics. One way to calculate this is to compute the geometric Berry phase $\gamma_B$ as one excitation encircles the other, as this phase is similar to two particle exchanges in terms of the spin-statistics theorem. The Berry phase is of the form [1].

$$\gamma_B = i \int dt \, \langle \psi(t) | \partial_t | \psi(t) \rangle \tag{19}$$

Since we are considering particle exchange, we will be considering the quasi-hole wavefunction of two holes, located at $\zeta_b(t)$ and $\zeta_c$, where only the $b$ hole is path dependent (parametrized by time) as it encircles the $c$ hole.

$$\psi_2^{+} = [z_a - \zeta_b(t)][z_a - \zeta_c] \prod_{i<j} (z_i - z_j)^m e^{-\sum_{k=1}^{N} |z_k|^2/4\ell_B^2} \tag{20}$$

Substituting this wavefunction into Eq. (19) allows us to write the Berry phase as

$$\begin{aligned} \gamma_B &= i \int dt \, \langle \psi(t) | \partial_t \zeta_b(t) | \psi(t) \rangle & (21) \\ &= i \int dt \int d^2z \; \partial_t [\ln(z - \zeta_b(t)] \psi_2^{+}(z,t) \delta(z - z_a) \psi_2^{+}(z,t) & (22) \\ &= -2\pi \int d^2z \; \rho & (23) \end{aligned}$$

where $\rho$ is a number density of the charge carriers [1]. From the form of the quasi-holes, one can see that it resembles a particle moving around a solenoid (Appendix A) [8]. With that picture in mind, one can see that encircling the quasi-hole solenoid gives us the flux quanta fraction.

$$\gamma_B = -\frac{2\pi}{m}\frac{\Phi}{\Phi_0} \tag{24}$$

where $\Phi_0$ is the flux quanta defined in Appendix A. Note that there is an extra factor of $m$ since the quasi-holes have charge fractional charge. Encircling the quasi-hole $c$ once corresponds to one flux quanta which implies that the change in the phase is [1]

$$\Delta\gamma_B = -\frac{2\pi}{m} \tag{25}$$

Physically this encircling should look no different than a double exchange of the two quasi-holes. However, there is non-vanishing phase (modulo $2\pi$) as we would suspect from either bosons or fermions, which suggests that these quasi-holes obey *fractional statistics*. These kinds of particles are known as *anyons*. From this analysis, one can see that quasi-holes pick up a phase of $e^{i\pi/m}$ under exchange and quasi-particles pick up a phase of $e^{-i\pi/m}$.

# Hall Conductivity Hierarchy

Despite predicting the emergence of fractional charge and fractional statistics, the Laughlin wavefunction is ultimately incomplete if it does not reproduce the Hall conductances. Fortunately, not only does it predict the filling fractions in which the conditions are valid, (i.e. $m$ must be an odd integer), with small modifications, it can predict even more filling fractions. We can first see how the conductance rises from the same scenario in Appendix A. As one increases the flux by one flux quanta, one increases the degeneracy, $m$. Recall that the LLL wavefunction (and by connections, the Laughlin wavefunction) is peaked at a circle of radius

$$R_{\max} = \sqrt{2m\ell_B^2} \tag{26}$$

which means that over time, if one continues adding flux quanta, charged particles will continue to move away radially [8], creating a current of

$$I_{xy} = \frac{e}{mT} \tag{27}$$

for charge carriers with arbitrary fractional charge $\pm\frac{e}{m}$ and for arbitrary time scale $T$. In that same we also slowly (frequency much smaller than the cyclotron frequency) increase the flux by one flux quanta.

$$V_{xy} = \frac{h}{eT} \tag{28}$$

Dividng the two previous equations by each other reproduces the Hall conductivity for the conditions of the Laughlin wavefunction which is for odd integer values of $m$.

$$\sigma_{xy} = \nu \frac{e^2}{h} \tag{29}$$

It turns out one can reproduce even more filling fractions if one considers the Coulombic interactions between quasi-particle/quasi-holes, which can be thought of as forming their own FQHE. One can then write down a Laughlin wavefunction for the quasi-particles/quasi-holes but now with different degeneracies because the quasi-particles/quasi-holes are anyons, not fermions [3].

$$\psi(z_i) = \prod_{i<j} (\zeta_i - \zeta_j)^{w \pm \frac{1}{m}} e^{-\sum_{k=1}^{N} |\zeta_k|^2/4\ell_B^2} \tag{30}$$

The term $w$ must be an even number due to the fact that the phase difference of $e^{i\pi/m}$ is invariant under addition modulo 2. Using similar arguments to that of Eq. (17) except with $\ell_B^2 \to m\ell_B^2$ since the charges of the quasi-particles/quasi-holes are $\mp \frac{e}{m}$ (recall that $\ell_B$ implicity has charge dependence in the denominator).

$$\mathcal{N} = \left( w \pm \frac{1}{m} \right) m^2 N \implies \nu = \frac{1}{wm^2 \pm m} \tag{31}$$

One performs this "FQHE within the FQHE" idea and for the $i$th iteration, the correction to the original Laugnlin filling factor becomes

$$\nu = \cfrac{1}{m \pm \cfrac{1}{w_1 \pm \cfrac{1}{w_2 \pm \ldots \pm \cfrac{1}{w_i}}}} \tag{32}$$

Whether or not we use the positive or negative sign depends on whether we are considering quasi-particles or quasi-holes. Out of all the plateaus in Fig. (1), the previous equation reproduces the $\nu = \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$ states. This is known as the Haldane-Halperin hierarchy of FQHE states [3]. As one can see from Fig. (1), this hiearchy is incomplete as the $\nu = \frac{2}{3}, \frac{3}{5}$ states, among others, are not accounted for. There are also states such as that are not shown in Fig. (1) such as the $\nu = \frac{1}{2}$ state. This can be explained using a composite fermion approach along with the presence of non-Abelian anyons [5]. For the sake of space, these topics will not be covered.

# Chern-Simons-Landau-Ginzburg Theory

So far we have only considered the FQHE from a bottom-up approach. While this does predict much of the emergent phenomena, some feel that it may be limiting especially given the fact that other phenomena such as superconductivity have had successful results from a top-down approach. The search for a Landau-Ginzburg style functional for the FQHE starts with a many-body Lagrangian for a particle in an electromagnetic field with Coulombic interactions [denoted as $V(x-y)$ for simplicity].

$$L_F = \frac{1}{2m}\sum_i \left| \left[-i\hbar\nabla - e\mathbf{A}(\mathbf{x}_i)\right]\psi\right|^2 - \sum_i eA_0|\psi|^2 - \int d^2x d^2y\ \delta\rho(x)V(x-y)\delta\rho(y) \tag{33}$$

where $\delta\rho(x) \equiv \psi^\dagger(x)\psi(x) - \langle\psi^\dagger(x)\psi(x)\rangle$, or the difference between the density operator and its average. Since theory is gauge invariant, one particular gauge we can choose is

$$A \to A + a = A + \theta\frac{e}{2\pi^2 h}\sum_{i\neq j}\nabla\alpha_{ij} \tag{34}$$

where $\alpha_{ij}$ is defined to be the angle of the vector connecting particles $i$ and $j$, and $\theta$ is a free parameter at the moment. This gauge transformation creates a new field, $\phi$ which differs from the old field, $\psi$ by a phase.

$$\phi(x_1,\ldots,x_N) = e^{-i\theta\sum_{i<j}\alpha_{ij}/\pi}\psi(x_1,\ldots,x_N) \tag{35}$$

The $\psi$ wavefunction (in the Hamiltonian picture) is fermionic so it is antisymmetric under the exchange of $i$ with $j$. How the $\phi$ wavefunction acts under exchange depends on the choice of $\theta$. Since the definition of $\alpha_{ij}$ implies that $\alpha_{ij} = \alpha_{ji} \pm \pi$, we can see that if $\theta = m\pi$ with $m$ odd, $\phi$ becomes symmetric under the exchange of $i$ with $j$. Thus one can interpret this gauge transformations as giving rise to composite bosons [10].

One can then write down the Lagrangian for these composite bosons. This Lagrangian has an extra term $a$ which can be interpreted as another gauge field.

$$L_B = \frac{1}{2m}\sum_i \left| \left[-i\hbar\nabla - e\mathbf{A}(\mathbf{x}_i) - e\mathbf{a}(\mathbf{x}_1)\right]\phi\right|^2 - \sum_i e(A_0 - a_0)|\phi|^2 - \int d^2x d^2y\ \delta\rho(x)V(x-y)\delta\rho(y) \tag{36}$$

This Lagrangian is similar to a theory of scalar electrodynamics in field theory which couples scalar bosons to a Maxwell term. One can then imagine coupling the previous Lagrangian to the corresponding Maxwell term for $\mathbf{a}$. This can be found by looking at the second quantized form of $\mathbf{a}$ (after taking $\nabla\alpha_{ij}$ in cylindrical coordinates).

$$a_\mu(x) = -m\frac{e}{2\pi h}\epsilon_{\mu\nu}\int d^2y\ \frac{x_\nu - y_\nu}{|x-y|^2}\rho(y) \tag{37}$$

where $\epsilon^{\mu\nu}$ is the antisymmetric tensor. Note that we are using the Einstein summation notation for repeated indices. If we act both sides with $\epsilon_{\alpha\mu}\partial_\alpha$ yields,

$$\epsilon_{\alpha\mu}\partial_\alpha a_\mu(x) = -m\frac{e}{2\pi h}\delta_{\alpha\nu}\partial_\alpha\int d^2y\frac{x_\nu - y^\nu}{|x-y|^2}\rho(y) \tag{38}$$

$$= m\frac{e}{h}\rho(x) \tag{39}$$

where the derivative gives a Dirac delta function. Taking the time derivative of both sides and applying the current continuity equation gives us

$$\epsilon_{\alpha\mu}\dot{a}_\mu = -m\frac{e}{\pi}j_\alpha \tag{40}$$

The above equation can be interpreted as an equation of motion for a Maxwell-like term with a current source. The Lagrangian for such a Maxwell-like term (without the source) is a Chern-Simons term [10].

$$L_a = \frac{h}{2em}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{41}$$

Since this Chern-Simons term along with the rest of the terms in Eq. (36) were constructed almost purely from the top-down approach like that of the Landau-Ginzburg theory for superconductivity, this theory is known as Chern-Simons-Landau-Ginzburg (CSLG) thoery,

The phenomenology of CSLG theory can be explored using mean-field theory and a convenient gauge choice. A particularly nice one is $a^\mu(x) = -A^\mu(x)$, which gets rid of everything except for the Chern-Simons term. The equation of motion for this gauge choice gives

$$\epsilon_{\mu\nu}\partial_\mu a_\nu = \Phi_0 m\bar{\rho} = B \tag{42}$$

with the $\Phi_0$ being the flux quantum defined in the Appendix A. The term $\frac{B}{\Phi_0}$ can be interpreted as a number density of a filled state, $\rho_A$ and so one can think of the filling factor as being the ratio between the average and the filled number densities.

$$\nu = \frac{\bar{\rho}}{\rho_B} = \frac{1}{m} \tag{43}$$

which reproduces the results of the Laughlin wavefunction. Fractional charge comes about when taking the integral of Eq. (42) over the area $\mathcal{A}$. Written in terms of vector notation, this becomes

$$\Phi_0 m\frac{\bar{q}}{e} = \int d\mathcal{A}\cdot\nabla\times\mathbf{a} = \oint d\mathbf{s}\cdot\mathbf{a} \tag{44}$$

where $\bar{q}$ is now an electric charge, not just a "number charge." Due to Maxwell's equations, the far RHS is just an integer multiple of the flux quantum, which implies again that when the RHS gives only one flux quanta, $\bar{q} = \pm\frac{e}{m}$ [10]. Using the Berry phase arguments gives the same fractional statistics as that of the Laughlin wavefunction. Likewise we can use the iterative process of "the FQHE within the FQHE," now with CSLG theory to reproduce the Haldane-Halperin hierarchy.

# Conclusion

While the FQHE already presents a long list of well-understood emergent phenomena such as anyons and fractional charges, there is still much to be done in terms of developing a more complete understanding of quantum Hall systems. A natural extension of the CSLG theory is to identify the order parameter. The mathematical form of the Chern-Simons term is found to be a topological, and thus the leading idea of the the kind of ordering that exists in the FQHE is topological order [9]. Perhaps coupled with the notions of topology, even more modern research has been trying to understand the geometric properties of the FHQE, both intrinsic (i.e. geometry due to the gauge group) and extrinsic (i.e. geometry due to a curved lattice).

# Appendix A: Flux Quanta and Angular Momentum

Imagine we have an infinitely long solenoid with a particle traveling along a ring outside the solenoid. From the definition of the flux, $\Phi$ the vector potential everywhere, where

$$A_\phi = \frac{\Phi}{2\pi r} \tag{45}$$

This is one physical scenario where the symmetric gauge is applicable. The Hamiltonian for such a system in cylindrical coordinates is

$$H = \frac{1}{2M}\left(-i\hbar\,\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{e\Phi}{2\pi r}\right)^2 \tag{46}$$

where $M$ in this case is the mass. The wavefunctions of this system can be solved to be

$$\psi = \frac{e^{im\phi}}{\sqrt{2\pi r}} \tag{47}$$

where $m$ can be thought of as an angular momentum term. Substituting this back into the Hamiltonian gives us

$$E = \frac{1}{2Mr^2}\left(\hbar m + \frac{e\Phi}{2\pi}\right) = \frac{1}{2Mr^2}\left(\hbar m + \frac{\Phi}{\Phi_0}\right) \tag{48}$$

where in the last equality we defined the flux quantum, $\Phi_0$ to be

$$\Phi_0 \equiv \frac{h}{e} \tag{49}$$

One thing is to note is that every time we increase by a flux quantum, we increase the angular momentum by one unit and thus increase the degeneracy [4,8].

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