Traffic Jams as Emergent Behavior

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Abstract: The flow of traffic can be thought of having phase transitions where macroscopic flow can be free, jammed or in an intermediate state between the two. Said transition can be seen to be first order. There are several models that describe the behavior of traffic ranging from the microscopic to macroscopic regime. One may even explain the system with the Ginzberg-Landau approach. Linear stability analysis reveals backwards propagating collective waves in many models of the system.

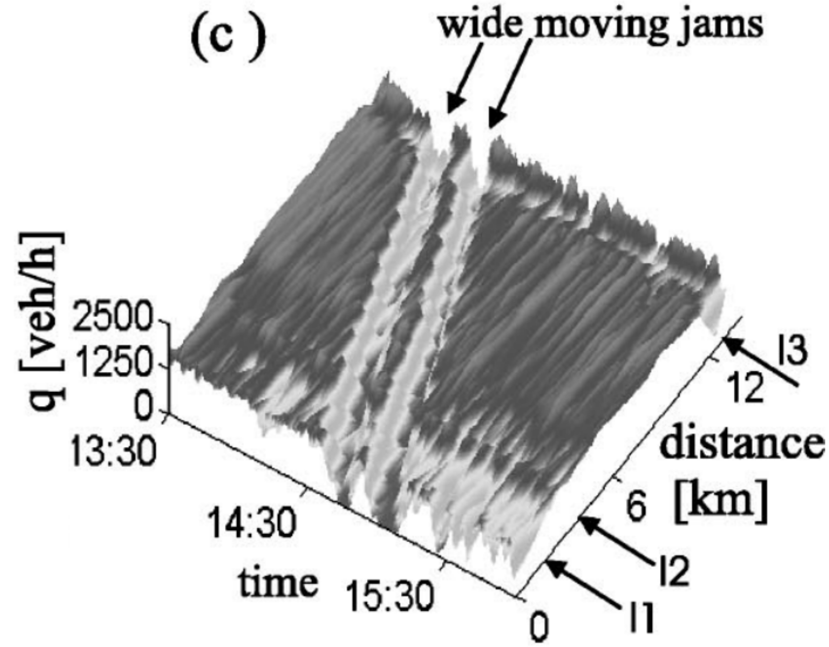
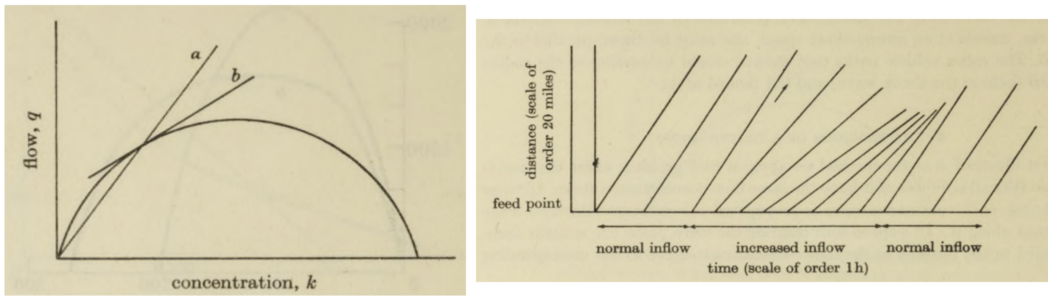


Figure 1: Spatio-temporal plot of traffic flow. Two jams can be seen propagating backwards through the road.[1]

Vehicular traffic has been studied as a fluid of interacting particles since the mid-20$^{\text{th}}$ century. Simple models were proposed to explain the sources of traffic jams, which seemingly appear out of nowhere on an individuals commute. Empirical evidence of traffic jams which propagate as collective waves in system of cars can be seen in Fig. 1. As personal automobiles became a predominant method of transportation for the average person, more studies of traffic systems surfaced along with an increase in computational power at the disposal of scientists. Using the same simple models from the first studies of traffic systems, newer computers could reveal obvious phase transitions and emergent behavior.

One of the earliest papers on traffic systems approached the problem on a macroscopic scale, looking at the flow of cars as one would a compressible one dimensional fluid. The flow must be some function of the density. The relationship obeyed



(a) (b)

Figure 2: Figures from Ref.[2]. (a) An example of the flow as a function of car density. (b) Each line represents the path of a car with varying flow rates, helping visualize the formation of traffic humps.

is [2]

$$v = \frac{j}{\rho}, \tag{1}$$

where j is defined as the number of vehicles which pass a segment of the road over a long period of time; a local current density. ρ is taken as the average value of the number cars on a segment of the road; a local car density. Therefore, v can be seen as a velocity which can vary with time and position on the road. From physical intuition, j as a function of ρ must increase at low values of ρ until it reaches some critical value where it begins to drop. An example of such a curve is shown in Fig.2a. In the language of propagating waves, v plays the part of the phase velocity, while the group velocity is given by

$$v_g = \frac{dj}{d\rho}, \tag{2}$$

An important prediction that can be made from this model is the formation of traffic jams where cars begin to drive slower. One can imagine the traffic system beginning in a homogeneous state where all cars travel at the same velocity, but a momentary increase in the flow will cause a traffic hump. This is represented as the bunching of wave fronts in Fig. 2b. The fact that v_g also decreases shows that the wave velocity of this collective behavior travels backwards down the road.

While the macroscopic fluid model is a popular way of looking at traffic systems, other models have been used to try and explain the emergence of traffic jams. For example, a Boltzmann-like approach can be taken to find all the important properties of the system. In Ref. [3], the velocity distribution function $f(x, v, t)dxdv$ is sought which gives the probability of finding a car in dx with a velocity in the range of dv at a time t. This serves as a kind of Maxwell-Boltzmann distribution function from kinetic gas theory. The equation that f is chosen to follow is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = (\frac{\partial f}{\partial t})_{relax} + (\frac{\partial f}{\partial t})_{collide}, \tag{3}$$

where $(\frac{\partial f}{\partial t})_{relax}$ is the restoring function that has f return to a steady state, and $(\frac{\partial f}{\partial t})_{collide}$ is the interaction term of two cars which try to avoid colliding with one another. An interesting fact about the terms on the right side of Eqn. (3) is that they incorporate the fact that cars can pass one another with some finite probability. Ultimately, Eqn. (3) is difficult to solve analytically. A simpler time independent solution found in Ref. [3] is

$$f = \frac{f_0}{1 - [c\tau(1-P)^2/P](v - \bar{v})}, \tag{4}$$

where f_0 is the distribution when there are no interactions present, P is the probability of a car passing another, τ is the relaxation time scale of $(\frac{\partial f}{\partial t})_{relax}$ such that

$$\frac{\partial f}{\partial t} = -\frac{(f - f_0)}{\tau(1-P)} \tag{5}$$

and $\bar{c(v)}$ a kind of average current density defined as

$$c\bar{v} = \int v f dv. \tag{6}$$

Although the time independent solution is quite simple, there are some important properties that can be gleaned from it. By setting P equal to 1, one is effectively making the interactions between cars vanish. Eqn. (4) shows that $f \to f_0$, which is makes physical sense. In the limit where P goes to 0, a non-trivial solution occurs only when $v \to \bar{v}$. Therefore, one can see a kind of condensation behavior occur where a large portion of cars decrease their velocity. This is similar to the emergence of traffic humps predicted in Ref. [3].

A popular small-scale model is the car following model, which states that the velocity of a car depends on the headway between itself and the preceding car. One of the earlier models introduced in 1961 is given by the equation [4]

$$v_j(t) = V_j - V_j exp(-\lambda_j V_j^{-1}[x_{j-1}(t-\tau) - x_j(t-\tau) - d_j]), \tag{7}$$

where v_j is the velocity of the j^{th} car, V_j is the maximum legal velocity of the road, τ is the time delay related to the reaction time of an individual driver, d_j is the minimum distance between drivers, and λ_j is the initial slope of the velocity. This is guessing that the velocity is an exponential growth function with respect to the headway, shown in Fig. 3.

By simplifying Eqn. (7) through various approximations, one can predict the emergence of waves similar to the traffic humps of [2], showing the continuum limit agrees with the car following model. However, it was near the turn of the century where it became more practical to use numerical methods to study this problem.

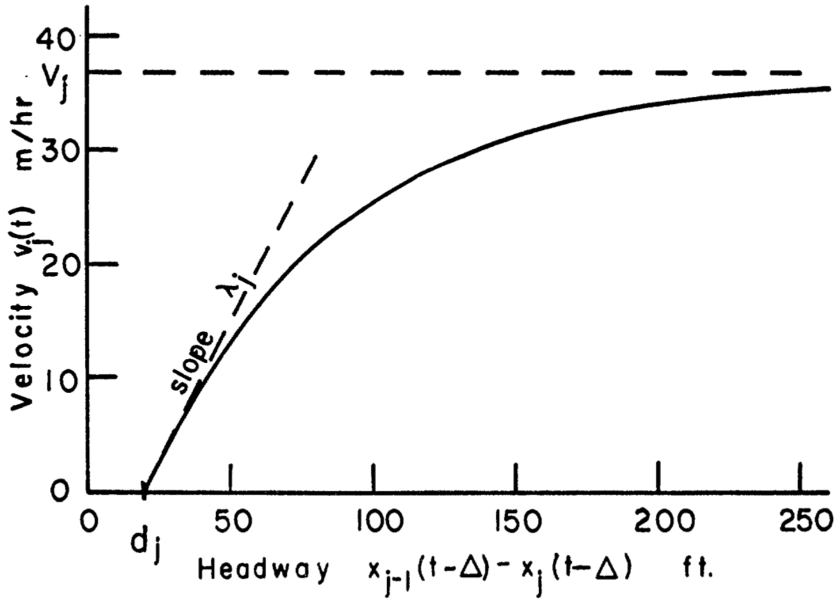


Figure 3: A simple model for the function of velocity of a car as a function of the headway between itself and the preceding car.[4]

Bando et. al[5] took a similar approach to the car following model, stating that

$$\ddot{x}_j = a[V(\Delta x_j) - \dot{x}]. \tag{8}$$

where a is the sensitivity of the function which depends inversely on the driver response time. The headway is denoted as $\Delta x_j = xj + 1 - x_j$, and $V(\Delta x_j)$ is the function that describes how fast a driver is going based on the headway. Similar to other papers, it was assumed that $V(\Delta x_j)$ must be monotonically increasing with a final maximum velocity. The steady state solution is very simple

$$x_j^0 = bj + ct, \tag{9}$$

where b is the length of the road (L) divided by the number of cars on the road (N) and $c = V(b)$ since all cars will be moving at a constant velocity with a constant headway. The stability of this state can be investigated by adding a perturbation y_j to x_j and equating the linear terms, which gives the equation

$$\ddot{y}_j = a[V'(b)\Delta y_j - \dot{y}]. \tag{10}$$

By taking the fourier expansion of Eqn. (10), one can find a solution numerically. To do this, set $y_k = exp[i\alpha_k + zt]$ with $z = u + iv$ (u and v are real) and set periodic boundary conditions to find discrete wave numbers such that $\alpha_k = 2\pi k/N$ ($k = 0, 1, 2...N - 1$). In doing so, a solution to y_k is proportional to e^{ut} and will grow exponentially in time if $u > 0$. However, for $u < 0$, the solution will decay over time and will return to the steady state solution. From Eqn. (10), the fourier expansion gives the equation

$$z^2 + az - aV'(b)(e^{i\alpha_k} - 1) = 0. \tag{11}$$

Expanding $e^{i\alpha_k}$ as $cos(\alpha_k) + isin(\alpha_k)$ and equating real and imaginary parts of Eqn. (10)

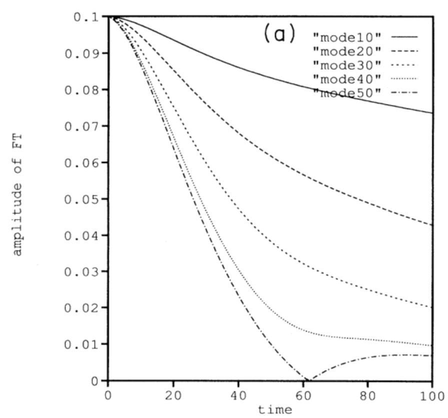
$$u^2 - v^2 + au - aV'(b)[cos(\alpha_k) - 1] = 0, \tag{12}$$

$$2uv + av - aV'(b)sin(\alpha_k). \tag{13}$$

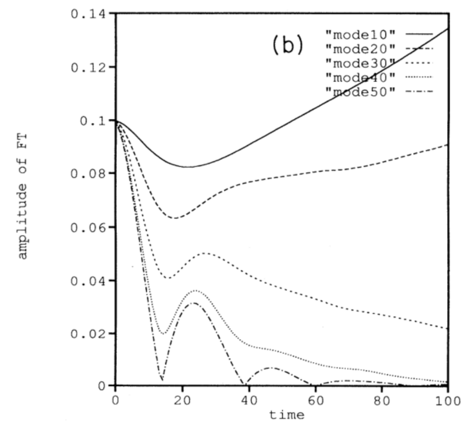
Looking at the solutions for $u = 0$, it can be seen that a curve in $V'(b) - \alpha$ space forms which separates phases of the traffic system into stable and unstable solutions. Said curve is written as

$$V'(b) = \frac{a}{2cos^2(2\alpha)}. \tag{14}$$

One can see that if $V'(b) < a/2$, then all solutions of will give $u < 0$ and the system is stable. By definition, a is inversely proportional to the reaction time of a driver to reach $V(\Delta x)$, so $a \rightarrow$ inf means a driver can react and change velocity



(a)



(b)

Figure 4: Figures from Ref.[5]. (a) Time evolution of fourier amplitudes with $V'(b) < a/2$ decay over time. (b) Time evolution of fourier amplitudes with $V'(b) < a/2$; some grow exponentially over time.

instantaneously, hence it makes physical sense that the system should always be in a stable state in this limit. When $V'(b) > a/2$, the system is predicted to leave the steady state once it evolves over time. This is demonstrated through numerical simulations shown in Fig. 4.

The numerical simulations in Fig. 4 were done with $V(\Delta x) = tanh(\Delta x - 2) + 2$, which satisfies all the requirements mentioned earlier while also having an inflection point. A plot of all the vehicle positions on the road over time is also shown in Fig. 5. It can be seen that traffic begins in a homogeneous state initially, but begins to form traffic humps which travel backwards down the road. This is what was predicted in the earliest papers, but is simpler to visualize with numerical simulations.

One can also take a step away from the more "microscopic" approach of the car following model to a thermodynamic model that utilizes the time dependent Ginzberg-Landau equation (TDGL). This is done in Ref. [6] utilizing many arguments similar

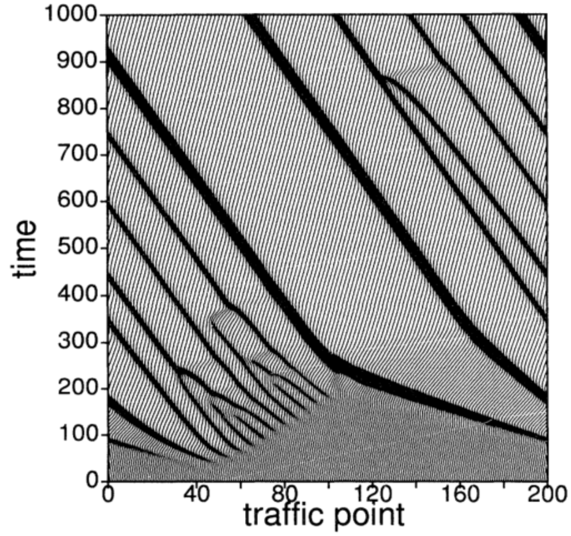


Figure 5: Darker spots represent the presence of a vehicle; thick black lines therefore indicate traffic jams which propagate backward over time.[5].

to papers discussed up to this point[4][5] to work with the equation

$$v_j(t\ +\ \tau)\ = (v_{max}/2)tanh(\Delta\ x\ -\ h_c)\ +\ h_c, \tag{15}$$

where h_c is the safety distance between cars. By looking at perturbations ($S(x,t)$) away from the steady state where all cars travel at equal velocities separated a distance h_c, a potential function can be derived, as well the TDGL for the system.

$$\partial_t S = [\partial_x - (1/2)\partial_x^2]\delta\Phi(S)/\delta S, \tag{16}$$

$$\Phi = \int dx[(V'/48)(\partial_x S)^2 - V(V'\tau - 1/2)S^2 + (|V'''|/24)S^4]. \tag{17}$$

Wanting to minimize Φ, we set the functional derivative $\delta\Phi/\delta S$ equal to 0. The homogeneous solution is given as

$$S(x,t) = \pm[6V'(2V'\tau - 1/2)/|V'''|]^{1/2}. \tag{18}$$

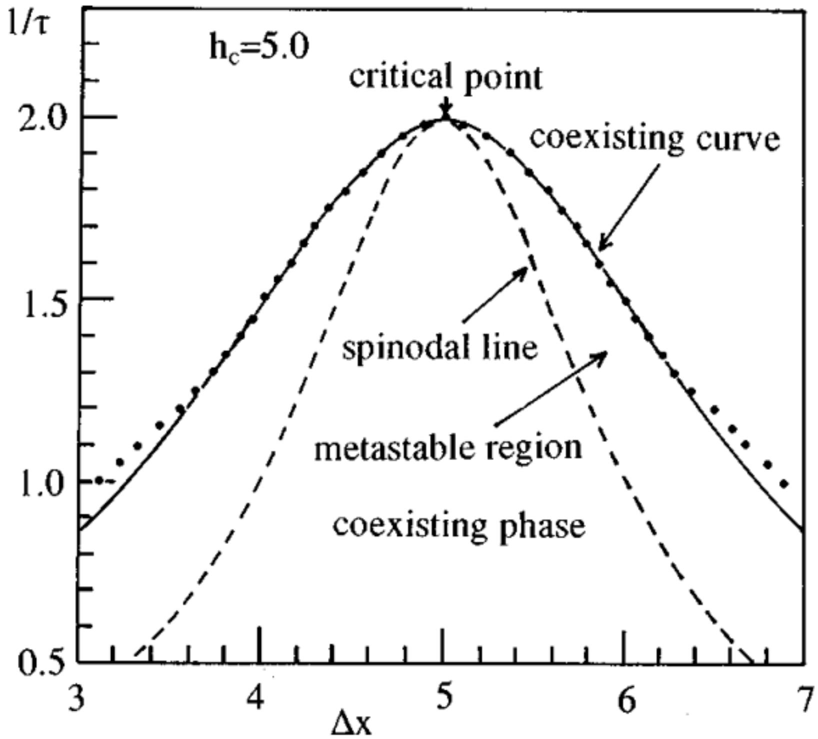


Figure 6: A phase diagram of a traffic system showing the separation of free flowing traffic and a jammed state.[6]

From the TDGL one finds that there is critical points of the phase diagram occur at $(\Delta\ x)_c\ h_c$ and $1/\tau_c\ =\ 2V'$. The phase diagram is shown in Fig. 6

Ultimately, one can see that both microscopic (microscopic on the scale of a large road) and macroscopic, coarse grained approaches agree with one another when describing a traffic system. A backwards propagating wave emerges due to the collective behavior of all the cars which interact with one another. Although this paper only covers a very small amount of examples, there are still other ways to tackle this problem[7]. One can introduce inhomogeneities in the system or use a cellular automata model, for example. In reviewing these papers, one can see the strength of linear stability analysis as well as the Ginzberg-Landau approach to studying the emergent behavior of traffic. However, due to society's increasing reliance on automobile transportation, studying more realistic models may help improve our understanding of the flow in highways and roads.

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