Synchronization of Chaos in Coupled Oscillators

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Abstract

This essay describes the experimental observation as well as numerical simulation and theory of synchronized chaos in systems of coupled oscillators. This work explores examples of this phenomenon in coupled electronic circuits and optoelectronic systems, as well as possible applications to secure communication.

1 Introduction and Background

The ever-present demand for secure wireless communication is one of the primary motivations for studying the synchronization of chaos. This demand, especially in military applications, led Louis Pecora and Thomas Carroll at the Naval Research Laboratory to develop a method for synchronizing two chaotic oscillators in 1990 [1], and it continues to be a reason for innovation in this field of study today [2]. When played through a speaker, the output of a chaotic system sounds like white noise [3]. But, unlike random noise coming from two unsynchronized and unrelated systems, the output of two synchronized chaotic oscillators is identical. Subtract the two and nothing but silence remains. The simplest way to apply this to improve the security of communications is to add a message signal to the "noise" from a chaotic oscillator before sending a transmission. Any eavesdropper will hear only seemingly random noise. The receiver, if equipped with an identical chaotic oscillator synchronized to the one in the transmitter, need only subtract the output of its chaotic oscillator from the transmission. The receiver will then have the message in its original form. Although this simple method for encrypting a signal is easily defeated by signal processing methods, synchronized chaotic oscillators continue to play a role in increasingly sophisticated methods for improving the security of communications [2, 4].

Synchronization of chaos applies to an extremely broad range of physical systems. Included are special electronic circuits, optical arrays, and even neurons and other biological systems [4]. Despite the widely different physical systems that exhibit chaos, it is often possible to write down simple coupled nonlinear equations to describe the behavior of the dynamical variables in the system. For example, the circuit shown in Fig. 4 is a modified Chua circuit. The time evolution of rescaled voltages x and y and a rescaled current z is described by Eq. 7 [2].

In general, synchronization of chaos requires more than simply preparing two identical systems with identical initial conditions. Since infinite precision in initial conditions is not possible to realize in practice, the dynamics of the two systems will at first track one another but will eventually fall out of step [3]. Synchronization of chaotic oscillators instead requires either feedback coupling, direct physical coupling, or a chaotic input signal. There are quite a few synchronization schemes that use these various methods; this essay will examine three. First, unidirectional linear error feedback coupling is currently the preferred method for inducing synchronization in two chaotic systems, even if they are not identical. Second, synchronization can also be induced by physical coupling, as in the case of parallel laser beams separated by a very small distance. Third, it is possible for sets of chaotic systems to synchronize when their parameters are driven by an external, different chaotic signal.

There are countless chaotic systems in nature as well as in the laboratory. To date there is no criteria that one can use to rule out the possibility of synchronized chaos in any of these. Rather, it is a matter of choosing two or more interesting systems, then finding a method to either couple them together or otherwise drive them in a way that produces synchronized chaos. Often it seems that chaotic systems prefer to synchronize; the theory given in Section 2.1 provides only the *minimum* linear error feedback required for stable synchronization.

2 Theory and numerical simulation

2.1 Unidirectional linear error feedback coupling

Since Pecora and Carroll's paper in 1990 [1], various feedback methods have been proposed as the best way to synchronize chaotic systems. In terms of simplicity and ease of physical implementation, one of the leading theories makes use of unidirectional linear error feedback coupling. Developed by Liu, et al. in 2002, this method is commonly used because it requires relatively low feedback gain and does not require the solution of nonlinear optimization problems [5]. This method was refined by Sun and Zhang [6], as well as Jiang, et al. [7]. Their work focuses on simple global synchronization criteria for coupled chaotic systems. This essay will follow Jiang, et al.'s treatment. Given a chaotic system in the form

$$\dot{x} = Ax + g(x) + u, \tag{1}$$

where x is the $n-$dimensional state vector, u is the $n-$dimensional input vector, A is an $n \times n$ matrix of constants, and $g(x)$ is a nonlinear, everywhere continuous function. We assume that $g(x)$ has a form such that

$$g(x) - g(\tilde{x}) = M_{x,\tilde{x}}(x - \tilde{x}), \tag{2}$$

where $M_{x,\tilde{x}}$ is a bounded matrix whose elements depend on x and $\tilde{x}$. We create a slave system based on Eq. 1:

$$\dot{\tilde{x}} = A\tilde{x} + g(\tilde{x}) + u + K(x - \tilde{x}), \tag{3}$$

where K is an $n \times n$ diagonal feedback matrix. By a judicious choice of the diagonal elements $k_i = K_{ii}$, one can ensure global asymptotic synchronization of the two systems. For a positive definite $n \times n$ matrix of constants P, if the k_i are chosen such that the eigenvalues λ_i of the matrix $Q \equiv \left[(A - K + M_{x,\tilde{x}})^T P + P(A - K + M_{x,\tilde{x}})\right]$ are all negative and bounded above by a constant $\mu < 0$, the two systems will be synchronized. Here we see the advantage of unidirectional linear error feedback coupling. The best choice of feedback is easy to calculate and implement physically, and the size of the feedback is necessarily on the order of the adjustable parameters in the system, A and M. The proof is explained in more detail in Appendix A, but the particular details are not crucial for understanding the material discussed in this essay. The key idea is that, given a system of equations that describes the dynamics of two or more chaotic oscillators, using the theory presented here will yield a feedback gain matrix K. This time-dependent matrix does not require difficult calculations at each time step and represents the minimum feedback required to induce synchronization between the two systems.

2.2 Synchronization via chaotic parameter driving

Synchronization via chaotic parameter driving is very different in character from the above methods of inducing synchronization in that it requires no coupling between the chaotic systems. A simple example of this phenomenon uses two uncoupled Lorenz systems (hereafter systems 1 and 2):

$$\begin{cases} x_i = \sigma_i(y_i - x_i), \\ y_i = \gamma_i x_i - y_i - x_i z_i, \quad i = 1, 2 \\ z_i = x_i y_i - b_i z_i, \end{cases} \tag{4}$$

where σ_i and b_i are chosen so that this systems is in the chaotic regime. The parameters γ_i are not fixed; instead we have

$$\gamma_i = d\,|p_n|\,, \quad i = 1, 2, \tag{5}$$

where d controls the driving strength and p_n is a different chaotic signal:

$$p_{n+1} = 1 - \mu p_n^2. \tag{6}$$

When systems 1 and 2 are started with nearly identical initial conditions and $d = 0$, their trajectories diverge. When they are started with very different initial conditions and $d \neq 0$, the influence of the chaotic input from Eq. 6 causes systems 1 and 2 to quickly synchronize. Fig. 1 shows the error $(x, y, z) \equiv (x_1 - x_2, y_1 - y_2, z_1 - z_2)$ between the dynamical variables of the two systems. This figure was generated by Guo-Hui Li using a fourth-order Runge-Kutta routine to perform a numerical simulation of the time evolution of the system given in Eq. 4 [8].

3 Experiment

3.1 Lasers

A laser is an example of a nonlinear oscillator whose behavior can be periodic or chaotic. It is possible to synchronize two lasers that are operating in a chaotic state. The experimental setup is very simple; two lasers are placed so that, after a series of mirrors and beam splitters, their beams are parallel and separated by a small distance d. The beam width is much less than a millimeter, and d ranges from about 0.5mm to 2.0mm. The beams themselves do not overlap, but for small enough d, the electric fields of the two beams overlap. This direct physical coupling of the two laser beams is enough to induce synchronization. For this system, synchronization means that the relative intensities of the two lasers track one another in time. Fig. 2 shows the relative intensities of the two beams, before and after synchronization [3].

The work of Wüunsche, et al. contains another example of direct physical coupling between laser beams [9]. As shown in Fig. 3, two identical semiconductor lasers are pointed so that their beams are collinear. The beams pass through collimators and a 50/50 beam splitter. Part of each beam goes to an oscilloscope, an electrical spectrum analyzer, and an optical spectrum analyzer, while the rest is injected into the opposite laser. This direct injection is the vehicle of physical coupling between the two lasers. Wünsche, et al. report synchronization similar to that of Fig. 2.

3.2 Electronic circuits

3.2.1 Two different circuits used to secure communication

There are many examples of experiments in which coupled chaotic oscillators synchronize [1, 2, 10]. The current standard procedure, given one or more chaotic electronic circuits, is to start with the nonlinear equations that describe certain voltages and currents as a function of time. Fotsin, et al. [2] begin with a modified Chua circuit (see Fig. 4). The time evolution of voltages and current in this circuit is given by Eq. 7:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1^3 + cx_1) + c_1 s(t), \\ \dot{y}_1 = x_1 - y_1 - z_1, \\ \dot{z}_1 = by_1, \end{cases} \tag{7}$$

where a, b, and c are constants. The additive term $c_1 s(t)$ consists of a scaling constant c_1 and an analog information signal $s(t)$ (the message).

This circuit was coupled to a different type of chaotic oscillator: a modified Van der Pol-Duffing oscillator. The time evolution of voltages and current in this circuit is given by Eq. 8:

$$\begin{cases} \dot{x}_2 = m(y_2 - x_2^3 + \alpha x_2 - \mu) - \tilde{k}_1(x_2 - x_1), \\ \dot{y}_2 = x_2 - y_2 - z_2, \\ \dot{z}_2 = \beta y_2 - \gamma z_2 - \tilde{k}_2(z_2 - z_1), \end{cases} \tag{8}$$

where x_2 and y_2 correspond to the voltages V_1 and V_2, rescaled and z_2 corresponds to I_L, rescaled. V_1, V_2, and I_L are indicated on the schematic of this circuit in Fig. 5. m, α, μ, β, and and γ are constants.

The next step is to determine the appropriate feedback gains $\tilde{k}_1$ and $\tilde{k}_1$ (as in Eq. 3) that ensure synchronization via unidirectional linear error feedback coupling, then implement the feedback in the electronic circuits. Using the technique presented in Section 2.1, Fotsin, et al. find that when the feedback gains are updated according to

$$\begin{aligned} \dot{\tilde{k}}_1 &= \gamma_1(x_2 - x_1)^2, \\ \dot{\tilde{k}}_2 &= \gamma_2(z_2 - z_1)^2, \end{aligned} \tag{9}$$

with $\gamma_1, \gamma_2 > 0$, (x_1, y_1, z_1) will asymptotically synchronize to (x_2, y_2, z_2). x_1 is then transmitted as an analog signal. To recover the message without differentiation of the transmitted signal, Fotsin, et al. use the following algorithm:

$$\begin{cases} \hat{s}(t) = c_1 k x_1 + w(t), \\ \dot{w}(t) = -c_1 k \left[(x_2 - y_2 - z_2) - \tilde{k}_1(x_2 - x_1) + c_1 \hat{s}(t) \right], \end{cases} \tag{10}$$

where $\hat{s}(t)$ is the recovered message and the gain $k > 0$ is a constant. Solving this system, one recovers $\hat{s}(t)$ (see Fig. 6) [2].

3.2.2 Chua circuit coupled to plasma discharge tube

Thus far we have examined experiments in which lasers were synchronized via direct physical coupling, and different types of electronic circuits were synchronized via linear error feedback coupling. In the last few years, researchers have been exploring the possibility of synchronization between a chaotic electronic circuit and an optoelectronic chaotic oscillator. For example, the work of Rosa, et al. involves the synchronization of a Chua oscillator circuit coupled to a plasma discharge tube [11]. In this experiment a voltage in the circuit synchronizes with the light intensity of the plasma discharge. The coupling is via unidirectional feedback; the voltage across C1 (see Fig. 4) is the driving signal for the plasma discharge tube. The evidence for synchronization is similar to that seen in Fig. 2. Despite the differences between these two chaotic physical systems, Rosa et al. were able to induce synchronization between them.

4 Discussion

Chaotic oscillators abound in both nature and in the physics laboratory, from nonlinear electronic circuits to lasers to natural chaotic systems such as global weather patterns or networks of neurons. The ability to induce synchronization between two chaotic systems is a big step toward exercising some control over their chaotic dynamics. It can also be the beginning of understanding the way that these systems work. Furthermore, once two chaotic oscillators are synchronized, it is not difficult to extend the same approach to a network of many such oscillators.

There are many applications for the the techniques presented in this essay. We have already seen a simple approach to secure communication; more sophisticated schemes exist that use synchronized chaotic oscillators. As another example, one could synchronize an array of chaotic lasers used in industrial manufacturing. If there are sixteen lasers, all cutting the same pattern, synchronizing their relative intensities will help guarantee that the parts produced are identical. Finally, there are biological applications. There is ongoing research in the synchronization of different cells for the purpose of treating disease [12]. If it is found that certain cells behave as chaotic oscillators, it may be possible to use feedback methods to induce synchronization.

Because of the broad range of chaotic oscillator systems that are susceptible to synchronization, this essay has been a survey of ongoing research topics in this field rather than a detailed discussion of a specific experiment or theory. We have discussed universal linear error feedback coupling theory; this is currently the standard method for synchronizing two or more chaotic oscillators that can be expressed as systems of coupled nonlinear equations. One advantages of this method is that it is computationally simple meaning, that there are no nonlinear systems to solve to determine the right feedback gain at each time step. Another advantage is that the feedback gains are necessarily on the order of the dynamical variables of the system. For example, if a voltage in a electronic chaotic oscillator ranges between 0 and 5 volts, the feedback gain will probably be of the same order of magnitude (less than 10 volts). Other methods of inducing synchronization require gains several orders of magnitude greater than those of the dynamical variables in the system [5]. While effective, this can be difficult to implement in a delicate electronic circuit. It is also inelegant in the sense that it is simply using brute force to overpower the natural dynamics of the system.

In this essay we have looked at another method to induce synchronization: driving one of the parameters in two chaotic systems with a different chaotic signal. Despite different initial conditions the two systems synchronize quickly in this setup. This is a relatively new area of research (2005); this method may prove useful in situations when feedback is difficult to implement in hardware [8]. We have also examined one of the many instances of synchronization of two different chaotic oscillator circuits via universal linear error feedback coupling. Our example was a modified Chua oscillator coupled to a modified Van der Pol-Duffing oscillator; this particular system included a direct application to secure communication [2]. Finally, we have discussed the synchronization of chaotic laser beams via direct physical coupling. Two methods were separating two parallel beams by a small distance so their electric fields overlap and coupling by directly injecting a laser beam into the resonant cavity of another laser. Both methods effectively produce synchronization without the need for delicate time-dependent feedback circuits. We also mentioned the coupling of a Chua oscillator circuit to a plasma discharge tube. In this experiment a voltage in the circuit synchronizes with the light intensity of the plasma discharge. Even though these are very different chaotic physical systems, it is nonetheless possible to induce synchronization between them [11]. Because of the wide applicability of synchronized chaos, it is likely that researchers will discover synchronization between even more exotic systems in the future. Synchronization of biological systems may yield unexpected but very useful applications, particularly in the field of medicine.

A Appendix: Exponential Lyapunov Stability

The notation in this Appendix follows that of Section 2.1; the theory presented in this section follows Jiang, et al. [7]. Lyapunov stability theory is the standard tool used to calculate the feedback required for synchronization [2, 10, 7, 5, 6]. First we define the error between the two systems as $e \equiv x - \tilde{x}$. For two chaotic systems global asymptotic synchronization means that the error e is globally exponentially stable about the origin. Lyapunov stability theory states that for exponential stability, $\dot{V} \leq 0$, where

$$V \equiv e^T P e, \tag{11}$$

where P is a positive definite matrix of constants and e is the error defined above. Since e and e^T both depend on time,

$$\dot{V} = \dot{e}^T P e + e^T P \dot{e}. \tag{12}$$

From Eqns. 1 and 3 we calculate:

$$\dot{e} = \dot{x} - \dot{\tilde{x}} = (A - K)e + g(x) - g(\tilde{x}) \tag{13}$$

After a few lines of algebra we obtain the result

$$\dot{V} = e^T \left[(A - K + M_{x,\tilde{x}})^T P + P(A - K + M_{x,\tilde{x}}) \right] e \equiv e^T Q e, \tag{14}$$

where $Q \equiv \left[(A - K + M_{x,\tilde{x}})^T P + P(A - K + M_{x,\tilde{x}}) \right]$. Given that the eigenvalues of Q are $\lambda_i, \;\; i = 1, 2, \ldots, n$, we can make a unitary transformation $Q = U^* \Lambda U$, where Λ is the diagonal $n \times n$ matrix with $\Lambda_{ii} = \lambda_i$. Thus our condition for stability $\dot{V} \leq 0$ becomes

$$\dot{V} = e^T Q e = e^T U^* \Lambda U e = e_1^T \Lambda e_1 \leq \mu e_1^T e_1 < 0, \tag{15}$$

where $e_1 \equiv Ue$. Provided that we choose the feedback gain K such that the eigenvalues of Q are negative and bounded above by by a negative constant μ, the two systems described by Eqns. 1 and 3 are globally asymptotically synchronized. A simple case is to use $P = I$, where I is the identity matrix. Then the condition for stability reduces to the expression

$$k_i \geq \frac{1}{2}(Q_{ii} + R_i - \mu), \quad i = 1, 2, \ldots, n, \tag{16}$$

where the Q_{ij} are the elements of Q and $R_i \equiv \sum_{j=1, j \neq i}^{n} |Q_{ij}|$. Note that this analysis applies even when the two systems are quite different; for example, Fotsin, et al. use this method to effectively induce synchronization between a Chua oscillator and a modified Van der Pol-Duffing oscillator [2].

References

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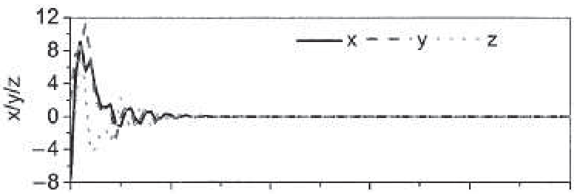

Figure 1: Complete synchronization for $d = 28$ and the initial condition $(x_1, y_1, y_1) = (1.5, 1, -5)$ and $(x_2, y_2, y_2) = (-8, 5, -9)$. Despite different initial conditions, the two systems synchronize quickly. Here $x \equiv x_1 - x_2$, $y \equiv y_1 - y_2$, and $z \equiv z_1 - z_2$. For this system, $\sigma_i = 10$, $b_i = 8/3$, $\mu = 1.95$, and $p_0 = 0.8$—all chosen such that the systems in Eqns. 4 and 6 are all in the chaotic regimes. Figure source is [8].

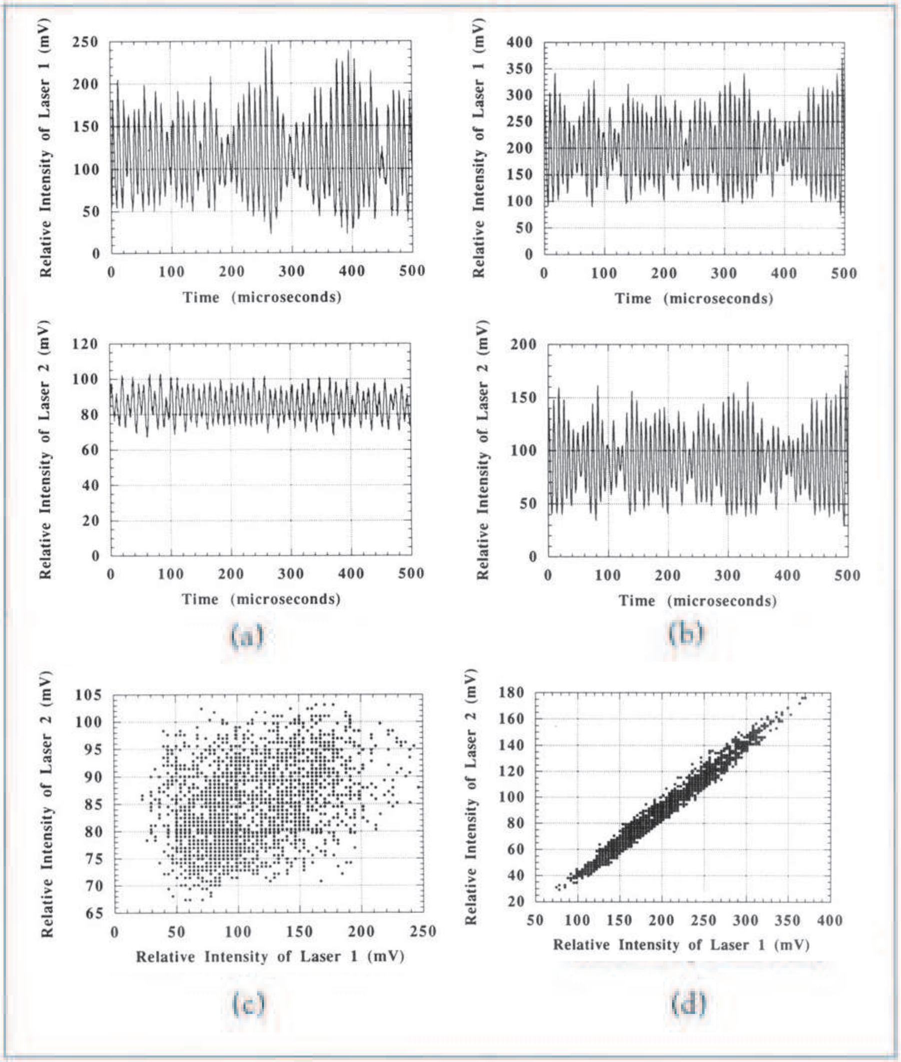

Figure 2: Synchronization of intensities for two coupled lasers. (a) shows the relative intensities of the two laser beams when they are separated by $d = 1.0$mm. The coupling is not strong enough to cause synchronization at this distance. (b) shows the relative intensities of the two laser beams when they are separated by $d = 0.75$mm. Here the coupling is strong enough to cause synchronization; the intensity versus time plots are nearly identical for both lasers. (c) shows the relative intensity of laser 1 plotted against the relative intensity of laser 2 at each point in time. Here $d = 1.0$mm and the two lasers are not synchronized. (d) shows the relative intensity of laser 1 plotted against the relative intensity of laser 2 at each point in time. Here $d = 0.75$ and the two lasers are synchronized. Figure source is [3].

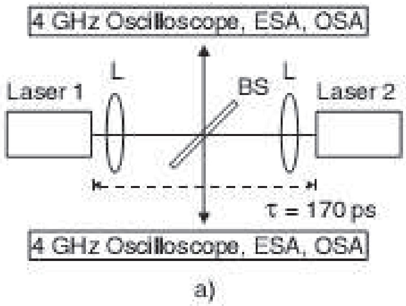

Figure 3: Experimental setup for two identical semiconductor lasers pointed directly at one another. (L) indicates a collimator and (BS) indicates a 50/50 beam splitter. Part of each beam goes to an oscilloscope for analysis, while the rest is injected into the opposite laser. This direct injection is the vehicle of physical coupling between the two lasers. Figure source is [9].

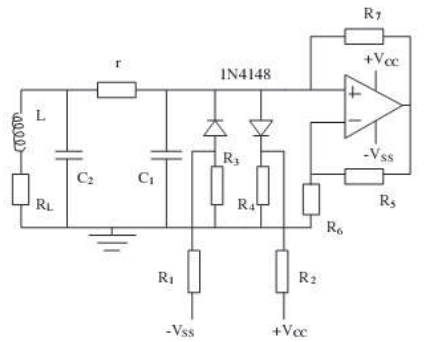

Figure 4: Schematic of a modified Chua circuit. The time evolution of voltages and current in this circuit is given by Eq. 7. Figure source is [2]; the specific values of the resistors, capacitors, diodes, and operational amplifiers in this circuit are given in this reference as well.

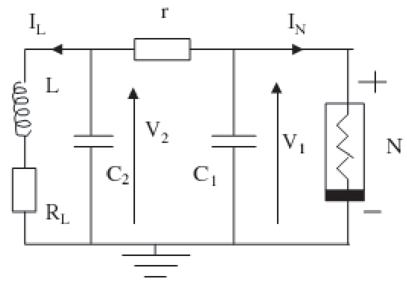

Figure 5: Schematic of a modified Van der Pol-Duffing oscillator circuit. The time evolution of voltages and current in this circuit is given by Eq. 8. Figure source is [2]; the specific values of the resistors, capacitors, diodes, and operational amplifiers in this circuit are given in this reference as well.

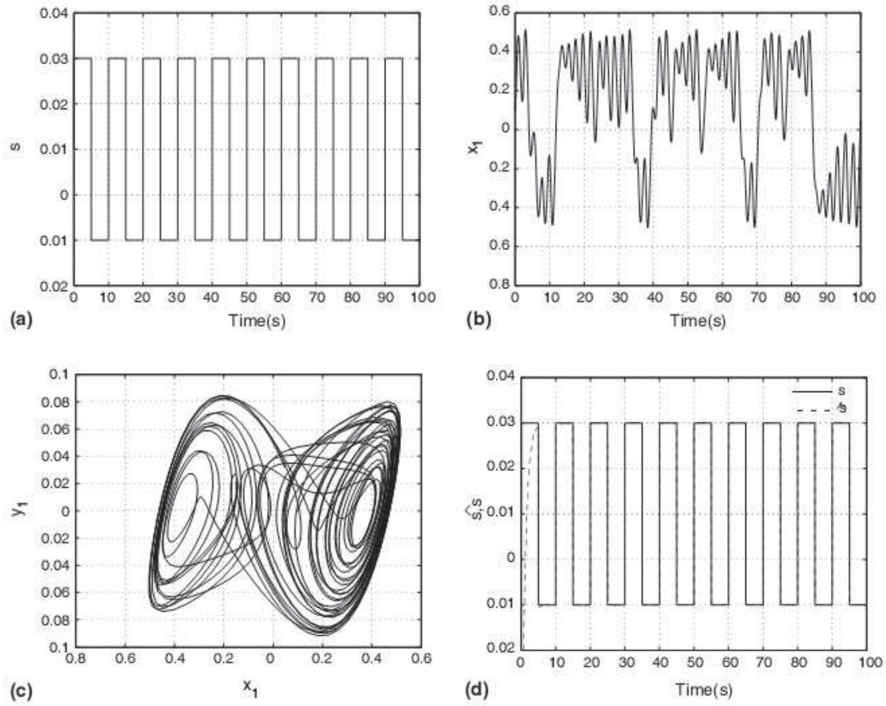

Figure 6: Behavior of the communication for a square wave message $s(t) = 0.01(1 + 2\,\mathrm{sign}(\sin \pi t/5))$. (a) Message signal, (b) transmitted signal x_1, (c) y_1 versus x_1, and (d) message signal $s(t)$ (solid line) and recovered signal $\hat{s}(t)$ (dashed line). Figure source is [2].