Spin transport in the square lattice vortex state of rotational ^{3}He-A superfluid

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We obtained a theoretical description of rotating superfluid in the ^{3}He-A phase in two dimensions. Experimentally, it is known that the vortices appear under rotation, so by requiring our order parameter to reflect that, we derived the form of the excitation eigenstates. Then we considered the presence of a magnetic field along the rotation axis, and found that the excitations evolve adiabatically with an extra phase (Berry phase) besides the dynamical one. Finally we considered the spin current carried by the excitations and found that it is quantized under certain conditions.

I. INTRODUCTION

^{3}He is among the rare type of systems that have the unique property of conserving quantum effects usually found only at microscopic scales when looking at it macroscopically. It has the property of remaining liquid even at absolute zero, and has several phases at low temperatures, in particular the ^{3}He-A phase around 3 mK. In these phases, the liquid enters a superfluid state, which supports topological defects. These defects are created spontaneously when the liquid is set to a rotation, creating different types of singularities, like point singularities, vortex lines and domain walls, as seen experimentally. In this essay we will describe a specific property of the A-phase when set in rotation, which is the spin transport of Bloch quasiparticles in the presence of a magnetic field, assuming a square lattice of line vortices.

The first part consists of deriving a Hamiltonian for the quasiparticle excitations of the ^{3}He-A phase in helium superfluid, using that the unbroken phase of ^{3}He has a symmetry group $SO(3)_S \otimes SO(3)_L \otimes U(1)_\omega$. Writing the most general form of the free energy to 4^{th} order in the order parameter, we get an expression for the order parameter, which we then use to motivate the form of the Hamiltonian for the excitations.

The second part consists in analyzing what happens when we set ^{3}He-A to a rotation in two dimensions. Above a critical velocity Ω_c, we get vortices that introduce a phase in the order parameter. This is shown in experiments in rotating ^{3}He-A, like the Finnish-Soviet R.O.T.A proyect [6]. For our calculations we assume a square lattice of vortices. In the presence of a magnetic field with a homogeneous gradient, we see that the evolution of the excitations is given by the usual dynamical phase plus a new phase (Berry phase), induced by the time dependent Hamiltonian, whose time dependency comes from the applied magnetic field. Finally, we use these results to calculate the Spin current carried by the excitations and find that it is quantized, and related to the Chern number of the n-th band.

II. DESCRIPTION OF SUPERFLUID ^{3}HE-A

The model that we use to describe superfluid ^{3}He is that of Cooper pairs, that is, the quasiparticles couple into pairs that form a non-interacting gas. Normal Cooper pairs possess no angular momentum around the center of mass, so by antisymmetry they must have quantum numbers S=0, L=0. In ^{3}He however, the pairs are in a p-wave state, so they have quantum numbers S=1, L=1.

This way, the symmetry group of ^{3}He in the unbroken phase must have, in the approximation of small spin-orbit coupling, a symmetry group given by

$$G = SO(3)_S \otimes SO(3)_L \otimes U(1)_\omega \tag{1}$$

Now that we know the symmetry group of ^{3}He, we set out to find the form of the order parameter. Writing the most general form on the free energy compatible with our symmetry group and minimizing it, we arrive at the form for the order parameter for the ^{3}He-A phase (see Apendix I for derivation):

$$A_{\alpha i} = \Delta_0 \hat{\mathbf{d}}_\alpha (\hat{\mathbf{x}}_i + i\hat{\mathbf{y}}_i)$$

where $\hat{\mathbf{d}}$ is the spin vector of the pair and $\hat{\mathbf{l}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$ is their orbital angular momentum. Following [1], we can see that the ^{3}He-A phase has a residual symmetry group H which is a subgroup of the original given by

$$H = SO(2)_S \times U(1)_{combined} \times Z_2 \tag{2}$$

To understand the origin of eq. (2), we see that we can still rotate around $\hat{\mathbf{d}}$, so that gives $SO(2)_S$. The second factor is the gauge transformation (global) U(1) with a parameter Φ together with a rotation about the $\hat{\mathbf{l}}$ axis by an angle θ_3 that matches Φ. Lastly, a spin rotation around an axis perpendicular to $\hat{\mathbf{d}}$ by an angle π can by compensated by a gauge transformation with $\Phi = \pi$, which yields the last factor.

Now that we've found the form of the order parameter for the ^{3}He-A phase, we seek to find a hamiltonian that describes the phase.

A. Hamiltonian for ^{3}He-A

We start by writing an action for non-relativistic fermions around the fermi energy that is compatible with the symmetry group of eq. (1)(the unbroken phase). We have

$$S = \int dtd^D x \psi^\dagger_\alpha(\mathbf{x})(i\partial_t - \epsilon(\hat{\mathbf{p}}))\psi_\alpha(\mathbf{x}) - \frac{\lambda}{2}\int dtd^D x \psi^\dagger_\sigma(\mathbf{x}) p_i \psi^\dagger_\tau(\mathbf{x})\psi_\tau(\mathbf{y}) p_i \psi_\sigma(\mathbf{y})$$

where $\psi_\alpha(\mathbf{x})$ is a Fermion field with spin $\alpha =\uparrow,\downarrow$, $p_i = -i\partial_i$ and $\epsilon(\hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2 - p_F^2}{2m}$

Introducing the auxiliary fields $\Delta^i_{\sigma\tau}$ and $(\Delta^\dagger)^i_{\sigma\tau}$ through the Hubbard-Stratonovich transformation to get rid of the quartic term, we get

$$S = \int dtd^Dx\psi^\dagger_\alpha(\mathbf{x})(i\partial_t - \epsilon(\hat{\mathbf{p}}))\psi_\alpha(\mathbf{x}) - \frac{1}{2}\int dtd^Dxd^Dy\left[\psi_\sigma(\mathbf{x})\Delta^\dagger_{\sigma\tau}(\mathbf{x},\mathbf{y})\psi_\tau(\mathbf{y}) + h.c.\right] \tag{3}$$

The auxiliary fields are directly related to the order parameter of ^{3}He-A. We want this action to describe the ^{3}He-A phase, and we still haven't specified how the auxiliary fields transform. Out of the original symmetry of eq. (1) that S satisfied, we should have, after spontaneous symmetry breaking, only the group given by eq. (2). This can be achieved by defining [1]

$$\Delta_{\sigma\tau}(\mathbf{x},\mathbf{y}) = \int \frac{d^Dp}{2\pi^D}e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}}\Delta_{\sigma\tau}(\mathbf{r},\mathbf{p})$$

Where $\mathbf{r} = (\mathbf{x}+\mathbf{y})/2$. The matrix $\Delta^i_{\sigma\tau}(\mathbf{r},\mathbf{p})$ is defined in terms of our order parameter

$$\Delta_{\sigma\tau}(\mathbf{r},\mathbf{p}) = A_{\alpha i}(\sigma^\alpha)_{\sigma\tau}i\sigma_y\mathbf{p}_i \tag{4}$$

where $A_{\alpha i}$ is the order parameter from the G-L theory we found before.

We can go from the Lagrangian given by eqn. 3 to the Hamiltonian through the usual Legendre transform $H = \dot{\psi}_\alpha\frac{\delta L}{\delta\partial_0\psi_\alpha} + h.c. - L$. We get

$$H = \int dtd^Dx\psi^\dagger_\alpha(\mathbf{x})\epsilon(\hat{\mathbf{p}}))\psi_\alpha(\mathbf{x}) + \frac{1}{2}\int dtd^Dxd^Dy\left[\psi_\sigma(\mathbf{x})(\Delta^\dagger)^i_{\sigma\tau}(\mathbf{x},\mathbf{y})\psi_\tau(\mathbf{y}) + h.c.\right]$$

Following [2], if we are in 2 dimensions and we make the angular momentum of all Cooper pairs point along the $\hat{z}$-axis, in the presence of a spin-orbit interaction $\hat{\mathbf{d}}$ will also point in the $\hat{z}$-axis. Then using eq. (4) we can write

$$\Delta_{\sigma\tau}(\mathbf{r},\mathbf{p}) = i\sigma_y(\sigma_3)_{\sigma\tau}\phi(\mathbf{r})(p_x + ip_y)$$

This way, the Hamiltonian for the ^{3}He-A phase in 2 dimensions is

$$H_{MF} = \int dtd^2x\psi^\dagger_\alpha(\mathbf{x})\epsilon(\hat{\mathbf{p}}))\psi_\alpha(\mathbf{x}) + \frac{1}{2}\int dtd^2xd^2y\left[\psi_\uparrow(\mathbf{x})\Delta^\dagger_A(\mathbf{x},\mathbf{y})\psi_\downarrow(\mathbf{y}) + \psi^\dagger_\uparrow(\mathbf{x})\Delta_A(\mathbf{x},\mathbf{y})\psi^\dagger_\downarrow(\mathbf{y})\right] \tag{5}$$

with $\Delta_A(\mathbf{x},\mathbf{y}) = 1/2\ \mathrm{Tr}[\sigma_x\Delta(\mathbf{x},\mathbf{y})]$

B. ^{3}He-A in a rotating cilinder

At T=0 and with the superfluid at rest, there won't be any excitations if we assume a finite energy gap in the spectrum. That's why we need to consider a cilinder of rotating superfluid ^{3}He-A, which for a velocity $\Omega > \Omega_{critical}$ will produce excitations and thus vortices. Under a rotation, the Hamiltonian transforms as

$$H_{Rot} = H_{MF} + \hat{\mathbf{L}} \cdot \hat{\mathbf{\Omega}}$$

where $\hat{\mathbf{L}}$ is the angular momentum. From this we obtain (without writing down the pairing terms)

$$H_{Rot} = \int dtd^D x \psi^\dagger_\alpha(\vec{\mathbf{x}})(\epsilon(\hat{\mathbf{p}}) - \hat{\mathbf{p}} \cdot (\vec{\mathbf{\Omega}} \times \vec{\mathbf{x}}))\psi_\alpha(\vec{\mathbf{x}}) = \int dtd^D x \psi^\dagger_\alpha(\vec{\mathbf{x}})[\epsilon(\hat{\mathbf{p}} - m(\vec{\mathbf{\Omega}} \times \vec{\mathbf{x}})) - \frac{m}{2}(\vec{\mathbf{\Omega}} \times \vec{\mathbf{x}})^2)]\psi_\alpha(\vec{\mathbf{x}})$$

In a typical experiment, we can neglect the $\frac{m}{2}(\vec{\mathbf{\Omega}} \times \vec{\mathbf{x}})^2$ term or introduce a parabolic trap [4]. This way, the hamiltonian for rotating superfluid turns out to be

$$H_{Rot} = \int dtd^D x \psi^\dagger_\alpha(\mathbf{x})\epsilon(\hat{\mathbf{p}} - m(\hat{\mathbf{\Omega}} \times \hat{\mathbf{x}}))\psi_\alpha(\mathbf{x}) + \frac{1}{2}\int dtd^D x d^D y \left[\psi_\uparrow(\mathbf{x})\Delta^\dagger_A(\mathbf{x},\mathbf{y})\psi_\downarrow(\mathbf{y}) + h.c.\right] \tag{6}$$

Now that the fluid is rotating, we see that for a given $\mathbf{p} > \mathbf{p}_F$ there will be a critical velocity Ω_c such that $\epsilon(\hat{\mathbf{p}} - m(\vec{\mathbf{\Omega}}_c \times \vec{\mathbf{x}})) = \epsilon(\hat{\mathbf{p}}_F)$, thus we'll get excitations.

C. Vortices and the vector potential

We see in eq. (6) that the rotation has induced a vector potential on the Hamiltonian much like an electromagnetic one would. For a general vector potential, we know that if we move around a curve in space, the fields are transformed by a phase given by the holonomy group associated with the vector potential [5]. In fiber bundle language, the vector potential is a connection on our real space that takes values on the gauge group, and the fields live on the associated bundle, which is a U(1) vector space. The gauge group are the rotations around the $\hat{z}$ axis which is $SO(2) \sim U(1)$. The holonomy group will depend on the topology of the fibre (for example, for a moebius strip it would be Z_2). In our case, we want our vector potential to induce vortices, so we impose that our holonomy group be Z (a vortex essentially is a topological defect, in the sense that when we go around it the field acquires a non trivial phase).

We have, for a general vector potential $\mathbf{A}(x)$, that the field transforms like

$$\psi(\Gamma(\vec{\mathbf{x}})) = D(e^{-i\int_\Gamma \vec{\mathbf{A}}\cdot\mathbf{d}\vec{\mathbf{\Gamma}}})\tilde{\psi}(\Gamma(\vec{\mathbf{x}})) \tag{7}$$

Where Γ is a general path in space, $\tilde{\psi}(\Gamma(\vec{\mathbf{x}}))$ is the gauge invariant part of the field (it's determined by fixing the gauge), and D is a representation of the gauge group SO(2) into the associated vector bundle U(1), which is just $D(e^{i\theta\sigma_3}) = e^{i\frac{\theta}{2}}$

Because the order parameter is of the form $\Delta_A \sim< \psi(x)\psi(y) >$, using eq. (7) we get the following form for the order parameter

$$\Delta_A(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = e^{-\frac{i}{2}[\oint_{\Gamma_x} \hat{\mathbf{A}}\cdot\mathbf{d}\hat{\mathbf{\Gamma}} + \oint_{\Gamma_y} \hat{\mathbf{A}}\cdot\mathbf{d}\hat{\mathbf{\Gamma}}]}\tilde{\Delta}_A(\hat{\mathbf{x}}, \hat{\mathbf{y}})$$

where $\tilde{\Delta}_A(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is the gauge invariant part (determined by fixing the gauge). If $\mathbf{A}(x)$ is integrable, then we get no phase shift and thus there are no vortices. Thus we want $\mathbf{A}(x)$ to have a net circulation, so we impose

$$\vec{\nabla} \times \vec{\mathbf{A}} = 2\pi \sum_i \delta^{(2)}(\vec{\mathbf{x}} - \vec{\mathbf{r}_i}) \tag{8}$$

This eq. represents a lattice of vortices of strength n=1, where the $\vec{\mathbf{r}_i}$ are the position of each vortex and form a square lattice[7] We can then write the order parameter as[8]

$$\Delta_A(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = e^{-\frac{i}{2}[\varphi(\vec{\mathbf{x}})+\varphi(\vec{\mathbf{y}})]}\tilde{\Delta}_A(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \tag{9}$$

So far, we haven't related eq. (8) to our actual vector potential, which is $\vec{\mathbf{A}}(\vec{\mathbf{x}}) = m(\vec{\mathbf{\Omega}}\times\vec{\mathbf{x}})$. Introducing this into eq. 8, we get

$$\vec{\nabla} \times \vec{\mathbf{A}} = 2m\vec{\mathbf{\Omega}} = 2\pi \sum_i \delta^{(2)}(\vec{\mathbf{x}} - \vec{\mathbf{r}_i}) \tag{10}$$

If we calculate the flux through a lattice cell of side a, we get the condition $a = \sqrt{\frac{\pi}{\Omega m}}$ for the lattice spacing (According to [6], for R=2.5mm and $\Omega = 1$rad/s there are approximately 600 vortex lines). With the help of eq. (9), we can write our hamiltonian as

$$H(\hat{\mathbf{p}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = \begin{pmatrix} \epsilon(\hat{\mathbf{p}} - m\mathbf{R})\delta(\vec{\mathbf{x}} - \vec{\mathbf{y}}) & \tilde{\Delta}_A(\vec{\mathbf{x}}, \vec{\mathbf{y}})e^{-\frac{i}{2}[\varphi(\vec{\mathbf{x}})+\varphi(\vec{\mathbf{y}})]} \\ -\tilde{\Delta}_A(\vec{\mathbf{x}}, \vec{\mathbf{y}})e^{-\frac{i}{2}[\varphi(\vec{\mathbf{x}})+\varphi(\vec{\mathbf{y}})]} & -\epsilon(\hat{\mathbf{p}} - m\mathbf{R})\delta(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \end{pmatrix} \tag{11}$$

where we introduce the Nambu representation, in which $\Psi \equiv (\psi_\uparrow, \psi_\downarrow^\dagger)^T$, and the complete Hamiltonian is given by

$$H_{rot} = \int d^D x d^D y \Psi^\dagger(\vec{\mathbf{x}}) H(\hat{\mathbf{p}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) \Psi(\vec{\mathbf{y}})$$

Following [3], we diagonalize H in the spin indices by introducing a Bogoliubov-de Gennes transformation, resulting in

$$\int d^2 y H(\hat{\mathbf{p}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) \Phi_E(\vec{\mathbf{y}}) = E \Phi_E(\vec{\mathbf{x}})$$

Where the $\Phi_E(\vec{\mathbf{x}})$ are linear combinations of the original fields. We can obtain properties of the eigenstates $\Phi_E(\vec{\mathbf{x}})$ looking at the symmetry of the vortex lattice. Following [2], we can define a translation operator given by

$$T_{\delta \mathbf{r}} = e^{i \delta \mathbf{r} \cdot (\hat{\mathbf{p}} + \vec{\mathbf{A}} \tau_3)} \tag{12}$$

where τ_3 is the third Pauli matrix in the Nambu (particle-hole) space[9]. It can be shown that the operators $T_{\mathbf{e}_x a}$ and $T_{\mathbf{e}_y a}$ commute with $H(\hat{\mathbf{p}}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ but not with each other. This non-commutativity comes from the fact that we are enclosing only one vortex in the translation path (see Apendix C), so we define new operators $T_{\mathbf{e}'_x d} \equiv T_{\mathbf{e}_x a + \mathbf{e}_y a}$ and $T_{\mathbf{e}'_y d} \equiv T_{\mathbf{e}_x a - \mathbf{e}_y a}$, where $d = \sqrt{2} a$. These operators satisfy

$$[H(\hat{\mathbf{p}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}), T_{\delta \mathbf{r}}] = [T_{\mathbf{e}'_x a}, T_{\mathbf{e}'_y a}] = 0$$

Therefore, the eigenstates of H are in the Bloch state (we can diagonalize H and T simultaneously), i.e.

$$\Phi_{\mathbf{k}}(\mathbf{x}) = e^{i \mathbf{k} \cdot \mathbf{x}} u_{\mathbf{k}}(\mathbf{x}) \tag{13}$$

Defining $H_{\mathbf{k}}(\mathbf{x}, \mathbf{y}) \equiv e^{-i \mathbf{k} \mathbf{x}} H(\hat{\mathbf{p}}, \mathbf{x}, \mathbf{y}) e^{i \mathbf{k} \mathbf{y}}$, the $u_{\mathbf{k}}(\mathbf{x})$ satisfy the eq.

$$\int d^2 y H_{\mathbf{k}}(\hat{\mathbf{p}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) u_{\mathbf{k}}(\vec{\mathbf{y}}) = E_{\mathbf{k}} u_{\mathbf{k}}(\vec{\mathbf{x}})$$

On the next section we will find a relationship between these Bloch states and the time dependent states that arise after introducing a magnetic field.

D. Rotating ^{3}He-A in a magnetic field

We want to study how the excitations are modified in the presence of a magnetic field. For this we introduce a field B with a homogeneous gradient in the $\hat{z}$ direction in the rotating frame, $B_z(\vec{\mathbf{x}}) = \vec{\mathbf{x}} \cdot \vec{\nabla} B_z$ (with $\vec{\nabla} B_z$ a constant vector). This field doesn't couple to the orbital angular momentum, only to spin through a Zeeman term. We can write

$$L_{spin} = \int d^2x \psi^\dagger_\alpha(\vec{\mathbf{x}})(\vec{\mathbf{S}} \cdot \vec{\mathbf{B}})_{\alpha\beta}(\vec{\mathbf{x}})\psi_\beta(\vec{\mathbf{x}}) = \int \frac{d^2x}{2}(\vec{\mathbf{x}} \cdot \vec{\nabla} B_z)\psi^\dagger_\alpha(\vec{\mathbf{x}})(\sigma_3)_{\alpha\beta}\psi_\beta(\vec{\mathbf{x}})$$

Using the Nambu notation, we can write this term together with the rest of the Lagrangian as

$$L = \int d^2x \Psi^\dagger(\vec{x}, t)\left(i\partial_0 - \frac{\vec{\mathbf{x}} \cdot \vec{\nabla} B_z}{2}\right)\Psi(\vec{x}, t) - \int d^2x d^2y \Psi^\dagger(\vec{x}, t) H(\vec{p}, \vec{x}, \vec{y})\Psi(\vec{x}, t)$$

We want to get rid of the spin term. For this, we notice that the time derivative can now be thought of as a covariant derivative $i\partial_0 - \frac{\vec{\mathbf{x}} \cdot \vec{\nabla} \mathbf{B}}{2} = i(\partial_0 + iA_0)$ with $A_0 = \frac{\vec{\mathbf{x}} \cdot \vec{\nabla} \mathbf{B}}{2}$. This way, we can make a gauge transformation so that A_0 will vanish in the new gauge. We have

$$A_0 \to A_0 - \frac{\partial}{\partial t}(t\frac{\vec{\mathbf{x}} \cdot \vec{\nabla} B_z}{2}) = A_0 - \frac{\vec{\mathbf{x}} \cdot \vec{\nabla} B_z}{2}$$

$$\vec{A} \to \vec{A} - \vec{\nabla}(t\frac{\vec{\mathbf{x}} \cdot \vec{\nabla} B_z}{2}) = \vec{A} - \frac{t\vec{\nabla} B_z}{2}$$

If we want the Lagrangian to be invariant under the transformation, the field must transform like

$$\Psi(\vec{x}, t) \to e^{-it(\frac{\vec{\mathbf{x}} \cdot \vec{\nabla} B_z}{2})\tau_3}\Psi(\vec{x}) \equiv \Psi'(\vec{x}, t)$$

Where τ_3 must be introduced to be consistant with the Nambu notation. Using the invariance of L under gauge transformations, in the new gauge we can write

$$L = \int d^2x \Psi'^\dagger(\vec{x}, t) i\partial_0 \Psi'(\vec{x}, t) - \int d^2x d^2y \Psi'^\dagger(\vec{x}, t) H(\vec{p} - \mathbf{f}(t), \vec{x}, \vec{y})\Psi'(\vec{y}, t) \qquad (14)$$

Where $\mathbf{f}(t) \equiv \frac{t\vec{\nabla} B_z}{2}$. We see that the in the new gauge, the Hamiltonian density operator is modified by a time dependent vector potential, induced from the spin potential A_0. The next step is to find the eigenstates of the new time dependent Hamiltonian, for this we must solve the time dependent Schroedinger eq.

$$i\frac{\partial}{\partial t}\Psi(t,\mathbf{x}) = \int d^2y H(t,\mathbf{x},\mathbf{y})\Psi(t,\mathbf{x})$$

where $H(t,\mathbf{x},\mathbf{y}) \equiv H(\vec{p}-\mathbf{f}(t),\vec{x},\vec{y})$. In the adiabatic approximation, the solution to the above equation is given by (see Apendix C)

$$\Psi_{\mathbf{k}}(t,\mathbf{x}) = e^{i\int_0^t dt^{'}(E_{\mathbf{k}}(t^{'}))+\gamma_{\mathbf{k}}(t^{'})}\Phi_{\mathbf{k}}(t,\mathbf{x}) \tag{15}$$

where $\Phi_{\mathbf{k}}(t,\mathbf{x})$ is an instantaneous eigenstate of the hamiltonian given by

$$\int d^2yH(t,\vec{\mathbf{x}},\vec{\mathbf{y}})\Phi_{\mathbf{k}}(t,\vec{\mathbf{y}}) = E_{\mathbf{k}}(t)\Phi_{\mathbf{k}}(t,\vec{\mathbf{x}}) \tag{16}$$

and $\gamma_{\mathbf{k}}(t^{'})$ is called the Berry phase, given by

$$\gamma_{\mathbf{k}}(t) = i\int_0^t dt^{'}\langle\Phi_{\mathbf{k}}(t^{'})|\frac{\partial}{\partial t^{'}}|\Phi_{\mathbf{k}}(t^{'})\rangle = i\int_0^t dt^{'}\langle u_{\mathbf{k}}(t^{'})|\frac{\partial}{\partial t^{'}}|u_{\mathbf{k}}(t^{'})\rangle \tag{17}$$

The $u_{\mathbf{k}}(t)$ are the Bloch time-dependent states, which are related to the time-independent ones in eq. (13) by $u_{\mathbf{k}}(t,\mathbf{x}) = u_{\mathbf{k}-\mathbf{f(t)}}(\mathbf{x})$. Thus eq. (15) tells us that the departure in time from an initial eigenstate of the Hamiltonian is given by the usual dynamical phase plus another phase of topological origin, the Berry phase.

E. Spin Hall effect

We know from eq. (2) that the ^{3}He-A phase has an $SO(2)_S$ symmetry even in the presence of a magnetic field. Using Noether's theorem, we can write a conservation law for the spin density, i.e., $\dot{\rho}^S + \nabla\cdot\mathbf{j^S} = 0$, where the spin density ρ^S is defined by $\rho^S(\mathbf{x}) = \frac{1}{2}\sum_{n\leq}\int_{BZ}\frac{d^2k}{2\pi}\Psi^{\dagger}_{n\mathbf{k}}(\mathbf{x})\Psi_{n\mathbf{k}}(\mathbf{x})$. The label 0 denotes the zero energy (wrt. the Fermi energy). In the presence of a uniform field $\mathbf{f}(t)$, the spin current is [2]

$$\langle\mathbf{j}^S(t)\rangle = \frac{i}{2}\sum_{n<0}\int_{BZ}\frac{d^2k}{2\pi}[\langle\dot{u}_{n\mathbf{k}}(t)|\frac{\partial u_{n\mathbf{k}}(t)}{\partial\mathbf{k}}\rangle - h.c.] = -\sigma^S_{xy}[\nabla B\times\mathbf{e}_z]$$

where

$$\sigma^s_{xy} = \frac{1}{8\pi}\sum_{n<0}\int_{BZ}\frac{d^2k}{2\pi i}[\nabla_{\mathbf{k}}\times < u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}} >]_z = \frac{1}{8\pi}\sum_{n<0}N^{(n)}_{Ch}$$

$N_{Ch}^{(n)}$ is the Chern number for the n-th band, and if we consider the case of no partially filled bands (We integrate over the complete Brillouin zone), it is a quantized number (See Conclusions). This way, we see that a non-zero current means we have excitations (quasi-particles) that carry spin with them.

III. CONCLUSION

In summary, we considered a particular phase of ^{3}He, the phase ^{3}He-A, which we obtained by spontaneous symmetry breaking of a general free energy functional[10] compatible with the symmetry group of ^{3}He determined experimentally. Once we obtained the Hamiltonian of the ^{3}He-A phase, we modified it by introducing a rotation in the fluid. Setting up this rotation creates vortices for a sufficiently high Ω , in particular we assumed a square vortex lattice. By looking at the group properties of translation operators we found that the excitation eigenstates form a band like structure. By introducing an aditional magnetic field along the axis of rotation of the fluid, we get a conductivity that depends on the curvature of the matrix elements of the time dependent Bloch states, which are related to the old ones by a shift in the momenta. This conductivity turns out to be quantized[11] if we sweep the whole Brillouin zone (which is a torus), and the value is given by the Chern Number for the n-th band.

IV. APPENDIX

A. Derivation of the order parameter

We start by writing the most general form of the free energy (Guinzburg-Landau potential) compatible with our symmetry group to 4^{th} order in parameter. Denoting our parameter by $A_{\alpha i}$, where α is a spin index, i a spatial index and A a complex number, the constraints we have are:

- $U(1)_\omega$ symmetry imposes that for every $A_{\alpha i}$ there has to be an $A^*_{\beta j}$. This implies no odd terms in F^{G-L}
- $SO(3)_S$ symmetry implies that every $A_{\alpha i}$ has to be contracted with an $A_{\alpha j}$ or $A^*_{\alpha j}$ so that it's invariant under rotations.

- $SO(3)_L$ Every $A_{\alpha i}$ has to be contracted with $A_{\beta i}$ or $A^*_{\beta i}$ for the same reasons as above

This way, the most general form of the free energy is (to 4^{th} order):

$$F^{G-L} = -\alpha A^*_{\alpha i}A_{\alpha i} + \beta_1 A^*_{\alpha i}A^*_{\alpha i}A_{\beta j}A_{\beta j} + \beta_2 A^*_{\alpha i}A_{\alpha i}A^*_{\beta j}A_{\beta j} + \beta_3 A^*_{\alpha i}A^*_{\beta i}A_{\alpha j} \;\; A_{\beta j} + \beta_4 A^*_{\alpha i}A_{\beta i}A_{\alpha j}A^*_{\beta j} + \beta_5 A^*_{\alpha i}A_{\beta i}A^*_{\alpha j}A_{\beta j}$$

We want to minimize F^{G-L} wrt. the order parameter. We have

$$0 = \frac{\delta F^{G-L}}{\delta A_{\mu\nu}} = -\tfrac{\alpha}{2}A^*_{\mu\nu} + \beta_1 A^*_{\alpha i}A^*_{\alpha i}A_{\mu\nu} + \beta_2 A^*_{\alpha i}A_{\alpha i}A^*_{\mu\nu} + \beta_3 A^*_{\mu i}A^*_{\alpha i}A_{\alpha\nu} + \beta_4 A^*_{\alpha nu}A^*_{\mu i}A_{\alpha i} + \beta_5 A^*_{\alpha nu}A^*_{\alpha i}A^{\mu i}$$

We see that if $\alpha < 0$ (above T_c) then $A_{\mu\nu} = 0$ (All terms are positive assuming all β's are positive). If $\alpha > 0$ then we can find nontrivial solutions. To simplify the notation, we can write the above eq. in matrix form

$$0 = -\frac{\alpha}{2}A^* + \beta_1 Tr(A^*A^\dagger)A + \beta_2 Tr(AA^\dagger)A^* + (AA^\dagger)^T(\beta_3 A + \beta_4 A^*) + \beta_5 AA^\dagger A^*$$

In general, values of A that differ by a phase transformation will represent the same phase, because of the U(1) symmetry. There are terms with A and with A^*, which won't mix between themselves under a U(1) transformation, so this tells us that the A and A^* terms represent different phases. We then set $\beta_1 = \beta_3 = 0$, which will turn out to represent the ^{3}He-A phase. The resulting eq for the order parameter is

$$-\frac{\alpha}{2} + \beta_2 Tr(\tilde{A}) + \beta_4 \tilde{A}^T + \beta_5 \tilde{A} = 0 \tag{18}$$

where $[\tilde{A}]_{\alpha j} = [AA^\dagger]_{\alpha j} = A_{\alpha i}A^*_{ji}$. We try with an expression of the form

$$A_{\alpha i} = \Delta_0 \hat{\mathbf{z}}_\alpha(\hat{\mathbf{x}}_i + i\hat{\mathbf{y}}_i) \tag{19}$$

where $\hat{\mathbf{x}}_i$, $\hat{\mathbf{y}}_i$ and $\hat{\mathbf{z}}_i$ are the components of the cartesian coordinate frame. Introducing into eq. (18), we get

$$-\frac{\alpha}{2}\delta_{\alpha,j} + 2{\Delta_0}^2\beta_2\delta_{\alpha,j} + 2{\Delta_0}^2(\beta_4 + \beta_5)\hat{\mathbf{z}}_\alpha\hat{\mathbf{z}}_j = 0$$

If we choose our coordinate frame to be the same as the spin frame, then we have $\hat{\mathbf{z}}_\alpha \hat{\mathbf{z}}_j = \delta_{\alpha,j}$. So we get

$$\Delta_0{}^2 = \frac{\alpha}{4(\beta_2 + \beta_4 + \beta_5)}$$

So we see that the expression in eq. (19) is valid if $\hat{\mathbf{z}}_\alpha$ is pointing along one of the spin axis. In a general basis, we'll have

$$A_{\alpha i} = \Delta_0 \hat{\mathbf{d}}_\alpha (\hat{\mathbf{x}}_i + i\hat{\mathbf{y}}_i) \tag{20}$$

where $\hat{\mathbf{d}}$ is the spin vector and $\hat{\mathbf{l}} = \hat{\mathbf{e_1}} \times \hat{\mathbf{e_2}}$ is the orbital angular momentum.

B. Commutator of the Translation Operators

We want to calculate

$$[T_{\mathbf{e}_x a}, T_{\mathbf{e}_y a}] = [e^{ia\mathbf{e_x}\cdot(\hat{\mathbf{p}}+\vec{\mathbf{A}}\tau_3)}, e^{ia\mathbf{e_x}\cdot(\hat{\mathbf{p}}+\vec{\mathbf{A}}\tau_3)}] = [e^{iA}, e^{iB}] \tag{21}$$

We also have

$$[iA, iB] = -i2ma^2\vec{\Omega}\cdot(\hat{e_x} \times \hat{e_y})$$

Because $[A,[A,B]] = [B,[A,B]] = 0$, we can use the Campbell-Baker-Haussdorf formula to write $[exp(iA), exp(iB)] = exp(i(A+B))sin(\frac{1}{2}[iA,iB])$. We see that we must make the sine vanish if we want the operators to commute. So

$$\sin(\frac{1}{2}[iA,iB]) = \sin(ma^2\Omega\tau_3) = \tau_3 \sin(\frac{ma^2\Omega}{2}) = 0$$

Using the value of the lattice spacing a found below eq.10, we get the condition $\pi = 2n\pi$ where n is an integer. This can't be satisfied, so the operators in eq. (21) don't commute. But we see that if we make a translation by a distance $d = \sqrt{2}a$, which is equivalent to enclosing 2 vortices, then the commutator will vanish. This is what we do when defining the operators $T_{\mathbf{e}'_x d}$ and $T_{\mathbf{e}'_y d}$ below eq. (12).

C. Adiabatic evolution of an eigenstate

We want to analize how the wavefunction of the system evolves in time if it starts at an eigenstate of the instantaneous Hamiltonian. For this we follow [5]. Let H be a hamiltonian that depends on a set of parameters $\mathbf{R}$ ($H = H(\mathbf{R}, \vec{x}, \vec{y})$). Note that in particular, the parameter can be time itself, as it'll be the case in our system. Let $|\mathbf{R}(t), t>$ represent the wavefunction of the system for a value of the parameters $\mathbf{R}(t)$. Assuming that the parameters describe a trayectory $\mathbf{R}(t)$ in parameter space, a guess for the wavefunction is

$$|\Psi(t)>= e^{[i\gamma_n(t) - i\int_0^t E_n(\mathbf{R}(t^{'}))dt^{'}]}|n, \mathbf{R}(t)> \tag{22}$$

Here we assume that the system is always in the n^{th} state (adiabatic assumption), which requires the Hamiltonian to be slowly varying in time ($|\nabla B_z| << 1$) (Slow variations in H means the perturbation has low frequency components only and cannot have enough energy to make a transition between states). Inserting this trial wavefunction into Schroedinger's equation, it's straightforward to show that

$$\gamma_n = \int_0^t < n, \mathbf{R}(t^{'})|\frac{\partial}{\partial t}|n, \mathbf{R}(t^{'}) >$$

In particular, choosing our parameter to be time itself (because $H = H(t, \vec{x}, \vec{y})$) we get

$$\mathbf{\Psi_k}(t, \mathbf{x}) = e^{i\int_0^t dt^{'}(E_\mathbf{k}(t^{'})) + \langle u_\mathbf{k}(t^{'})|\frac{\partial}{\partial t^{'}}|u_\mathbf{k}(t^{'})\rangle}$$

where $\mathbf{\Phi_k}(t, \mathbf{x})$ is an instantaneous eigenstate of the hamiltonian given by

$$\int d^2y H(t, \vec{\mathbf{x}}, \vec{\mathbf{y}})\Phi_\mathbf{k}(t, \vec{\mathbf{y}}) = E_\mathbf{k}(t)\Phi_\mathbf{k}(t, \vec{\mathbf{x}}) \tag{23}$$

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[7] Here we consider line vortices, but experiments [6] have shown that there are seven different types of vortices in the ^{3}He-A and ^{3}He-B phases. There are also other possible latices, see [6] for a table.

[8] Notice that the phases in eq. 9 are multivalued functions.

[9] Its physical meaning is that holes have opposite momenta than particles.

[10] Up to 4th order in the order parameter.

[11] The $A(\mathbf{k}) \equiv < u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}} >$ connection form, which is defined on the Brillouin zone torus, must be single valued when making a gauge transformation, which imposes a quantization condition.