Voting, a process of emergence of order

PHYS 569 Term Paper

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Abstract

Consensus in a vote is an emergence of order. In this term paper, voter models on lattice and graphs are reviewed, with particular emphasis on dimension and topology underlying the process of approaching consensus. Its found that when individuals are connected to few neighbors, like in 1D and 2D lattice, consensus can always be achieved in both finite and infinite systems; when individuals have many neighbors, like in high dimensional lattice and graphs, consensus can only be reached in finite systems. Time needed for consensus is calculated for each finite system discussed. Numerical experiments are also presented to supplement the theory.

1 Introduction

"How will you decide where to have group lunch?" "Vote by group members!"

Voting is what we will usually do when facing choices as a group, as it's a simple way to bring order out of opinion chaos. People are concerned about vote because the result matters to the way we live. Particularly, vote in politics gains special attention because it shapes the near future of a country.

Being a social event though, vote has interested both mathematicians and physicists. From the perspective of math, vote belongs to stochastic processes, which form an important category of problems in probability theory. Viewed physically, change of mind resembles flip of spin and voters form a spin system upon which a statistical ensemble can be produced.

Research on vote, especially two-choice vote, has lasted for almost 40 years. Clifford and Sudbury [1] firstly developed a model to describe the process of species invasion, which was in their context equivalent to a voting event. Holley and Liggett [2] then formally proposed the name "voter model".

The definition of voter model is neat: in a system consisting of a certain number (which can be infinity) of individuals, each individual holds an opinion, either A or B. Any individual is only influenced by its nearest neighbors and will randomly pick up the opinion of one of the neighbors. Individuals keep interacting with each other until consensus is realized, where everyone in the system shares the same opinion, either A or B. Basically, there're two questions to answer in voter model: Does there exist a consensus at all? How long will it take to realize consensus? The answers depend on system dimension and size. And the topology of the system also casts essential influence on the evolution of voting.

In this term paper, answers to the above two questions will be reviewed. The following sections are organized in this way: Section 2 focuses on the voter model on D dimensional hypercubic lattice, where each point has 2D neighbors; Section 3 discusses the voter model on graphs, where each node has an arbitrary number of neighbors; some modifications to voter model are introduced in Section 4; Section 5 concludes the paper.

2 Voter model on hypercubic lattice

Physicists have shown great interest in voter model, since it can be applied to describe phase transitions in the absence of interface tension (as will be clarified below) [3] and heterogeneous catalyst in reaction controlled limit [4]. Instead of using the random walk process to paraphrase voter model as mathematicians do, physicists interpret the voter model by Glauber model of spin lattice [5]. On a D dimensional hypercubic lattice, N nodes are each endowed a spin, with spin +1 and −1 representing opinion A and B respectively. The change of opinion then corresponds to a flip of spin.

The essential point of lattice voter model is to assign every individual an equal number of neighbors, which is equivalent to assume that each individual is equally connected to and familiar with its environment. This actually neglects much diversity in human society. It'll be shown that the model is valid qualitatively, yet too simple in a quantitative sense.

Let's denote the lattice points by $\vec{k} = (k_1, k_2, \ldots, k_D)$. $S_{\vec{k}} = \pm 1$ represents the spin at point $\vec{k}$. S stands for the spin configuration of the entire system. The probability of spin flipping at $\vec{k}$, $(S_{\vec{k}} \to -S_{\vec{k}})$, is

$$W_{\vec{k}}(S) = \frac{1}{\tau}\left(1 - \frac{1}{2D} S_{\vec{k}} \sum_{\vec{e}_i} S_{\vec{k}+\vec{e}_i}\right), \tag{1}$$

where τ is the time scale of the flipping and the summation is over the 2D nearest neighbors. τ was usually selected to be $\tau = 4/D$ for simplification [6].

Introduce a quantity $P(S,t)$ to denote the probability density of the spin configuration being S at time t. Then voting process is holographically described by the time evolution of $P(S,t)$. The master equation that determines $P(S,t)$ comes as follows:

$$\frac{d}{dt}P(S,t) = \sum_{\vec{k}} W_{\vec{k}}(S^{\vec{k}})P(S^{\vec{k}},t) - \sum_{\vec{k}} W_{\vec{k}}(S)P(S,t), \tag{2}$$

where $S^{\vec{k}}$ is different from S only at $\vec{k}$. The first term in the above equation accounts for spin flipping to form configuration S, and the second term for spin flipping to deviate S. Based on the above equation, the equation of motion for correlation function of spin $\langle S_{\vec{k}} \ldots S_{\vec{l}} \rangle = \sum_S S_{\vec{k}} \ldots S_{\vec{l}} P(S,t)$, up to any order, can be obtained.

The first and the second order, or the single-body and two-body, correlation functions are of special interest. For the single-body correlation function [6],

$$4\frac{d}{dt}\langle S_{\vec{k}} \rangle = \Delta_{\vec{k}} S_{\vec{k}}, \tag{3}$$

where $\Delta_{\vec{k}}$ is the discrete Laplace operation

$$\Delta_{\vec{k}}\langle S_{\vec{k}}\rangle = -2D\langle S_{\vec{k}}\rangle + \sum_{\vec{e}_i}\langle S_{\vec{k}+\vec{e}_i}\rangle.$$

From the differential equation (3), it's straightforward to get that

$$\frac{d}{dt}\sum_{\vec{k}}\langle S_{\vec{k}}\rangle = 0. \tag{4}$$

This means the magnetization $\langle S\rangle = (\lim_{N\to\infty})\, 1/N \sum_{\vec{k}} S_{\vec{k}}$ is conserved. Furthermore, the consensus, if can be realized, is determined by the initial magnetization.

The two-body correlation function satisfies the following differential equation [6]

$$4\frac{d}{dt}\langle S_{\vec{k}}S_{\vec{l}}\rangle = \left(\Delta_{\vec{k}} + \Delta_{\vec{l}}\right) S_{\vec{k}}S_{\vec{l}}. \tag{5}$$

The above equations hold for both finite and infinite lattices. Skipping the tedious derivations, results of voting in finite and infinite lattices are presented directly here.

For a finite lattice with N points, consensus can always be obtained no matter what the dimension is. However, the time cost to realize consensus depends on the dimension as well as on the size of the lattice. According to the work by Cox [7], the consensus time scales with N in the following way

$$T_N \sim \begin{cases} N^2, & d=1 \\ N\ln N, & d=2 \\ N, & d>2 \end{cases} \tag{6}$$

Since the lattice voter model simplifies the real human connection too much, the resulting T_N above may not be a good fitting to the real voting process.

For an infinite lattice, Frachebourg and Krapivsky [6] solved the voter model problem analytically in an equivalent situation, the heterogeneous catalyst. In their work, the differential equation (5) with respect to the two-body correlation function was solved by Laplace transform. The density of interface between points bearing different spins is related to the two-body correlation function in the sense that $n_{AB}(t) = \left(1 - \langle S_{\vec{k}}S_{\vec{k}+\vec{e}_i}\rangle\right)/2$. And the asymptotic behavior of n_{AB} as a function of time comes as

$$n_{AB}(t) \sim \begin{cases} t^{-1+D/2}, & D < 2 \\ (\ln t)^{-1}, & D = 2 \\ a - bt^{-D/2}, & D > 2 \end{cases} \quad t \to \infty \tag{7}$$

where a and b are non-zero constants. It's concluded from the above equation that for infinite 1D and 2D lattice, $n_{AB} \to 0$ as $t \to \infty$ and consensus is realized; however, for higher dimensional infinite lattice, $n_{AB} \to a \neq 0$ as $t \to \infty$ and the system takes on a state where both opinions coexist.

Specially, in 2D lattice, the explicit form of n_{AB} is [6]

$$n_{AB}(t) = \frac{\pi}{2\ln(t) + \ln(256)} + \mathcal{O}\left(\frac{\ln t}{t}\right). \tag{8}$$

Since the denominator contains a non-negligible term for small t, the time span must be long enough in order to observe the asymptotic behavior of $n_{AB} \sim (\ln t)^{-\sigma}$ with $\sigma = 1$ in numerical experiments. Actually, people did not get the -1 exponent but $\sigma = 0.59$ in the beginning [4], due to too short simulation time t and too small lattice size. Frachebourg and Krapivsky performed the simulation with a much larger lattice and found that $\sigma \approx 0.51$ for $t \leq 1500\tau$ and $\sigma \approx 0.67$ for $t \leq 10^4\tau$ [6]. They thus believed that $\sigma \to 1$ asymptotically as $t \to \infty$.

The theoretical analyses and numerical experiments on lattice voter model above tell us that in 1D and 2D lattices, consensus is always realized no matter how large the system is; however, in higher dimensions, consensus is only obtainable in finite systems. This means whether an agreement can be reached among a community is heavily affected by the connection between individuals within. It's always possible to get people organized to one opinion in a society where each person is limited to only few acquaintances; while, in a more sophisticated society where people are more tightly bonded, it's impossible to wash their minds. This reminds me of the dream of Lao Tzu, more than two thousand years ago, of having a small country with few people isolated from each other in order for good governance. It's mathematically true that smaller isolated population is easier to govern, and the society as a whole gets confused when individuals are open to many voices around.

Another interesting point about the lattice voter model to demonstrate before closing the section is that the above model can be applied to simulate phase transitions with no interface tension. To manifest this, let's consider a circular area of spins up surrounded by spins down in 2 dimension. As time goes by, the radius of the circle stay invariant statistically, implying a zero interface tension, as illustrated in Figure 1 [3].

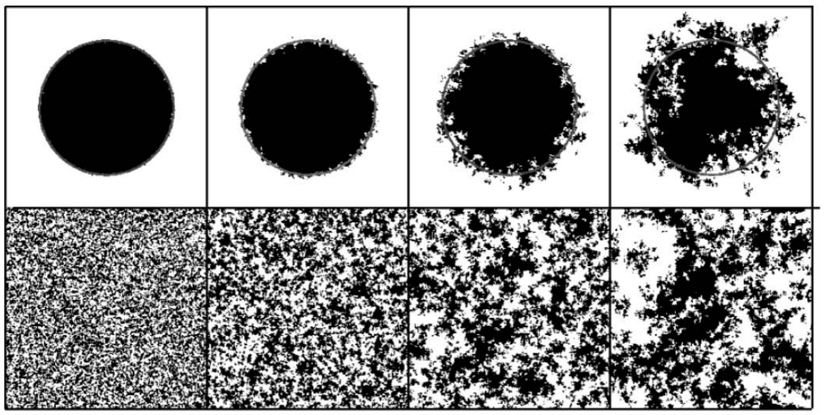

Figure 1: Illustration of the domain growth in the D=2 voter model (system size 256^2). Top: Snapshots at times t=4, 16, 64, 256 during the evolution of a bubble of initial radius $r_0 = 180$ (thin circle). Bottom: same from symmetric ric. (Figure reproduced from [3])

3 Voter model on network

As has been pointed out before, the lattice voter model assumes that everyone has an equal number of neighbors. This assumption is too simple for the complicated social network, where it's always the case that some people have more friends than others. To describe such a network precisely, the voter model based on graph theory is developed. We'll still assume that the influence of each neighbor has the same weight, and explore the variation of voting result due to the change in topology of the network from lattice to graph.

In the language of graph theory, the number of neighbors of a node is called the degree of that node. In a finite size graph consisting of N nodes, the fraction of nodes having degree k is $n_k \equiv N_k/N$. n_k obeys certain distribution with respect to k. Then $\mu_1 \equiv \sum_k kn_k$ is the average degree of each node, or the first moment of the initial degree distribution; $\mu_2 \equiv \sum_k k^2 n_k$ is the second moment of the initial degree distribution. Define the average degree-weighted density of up spins, ω, for the initial graph as follows [8]

$$\omega \equiv \frac{1}{\mu_1 N} \sum_{\substack{y \\ S_y=+1}} k_y, \tag{9}$$

where y in the summation stands for the node in the graph and k_y is the degree of node y. It follows that

$$1-\omega = \frac{1}{\mu_1 N} \sum_{\substack{y \\ S_y=-1}} k_y. \tag{10}$$

The time needed to reach consensus depends on the initial distribution and the number of nodes N. It satisfies the following differential equation [8],

$$\frac{1}{N}\frac{\mu_2}{\mu_1^2}\omega(1-\omega)\partial_\omega^2 T_N = -1. \tag{11}$$

The boundary condition for the above equation is

$$T_N(\omega=0)=0, \quad T_N(\omega=1)=0. \tag{12}$$

The solution is then [8]

$$T_N(\omega) = -N\frac{\mu_1^2}{\mu_2}\left[(1-\omega)\ln(1-\omega)+\omega\ln\omega\right]. \tag{13}$$

It's important to notice that the mth moment $\mu_m \sim \int^{k_{max}} k^m n_k dk$.

To gain a view into the form of T_N, we can consider a power law degree distribution, $n_k \sim k^{-\nu}$. k_{max} is related to N in the sense that $\int^{k_{max}} n_k dk = 1/N$, therefore $k_{max} \sim N^{1/(\nu-1)}$. Putting the moments as functions of N together, T_N is found to be [8]

$$T_N \sim \begin{cases} N, & \nu > 3 \\ N/\ln N, & \nu = 3 \\ N^{(2\nu-4)/(\nu-1)}, & 2<\nu<3 \\ (\ln N)^2, & \nu = 2 \\ \mathcal{O}(1), & \nu < 2. \end{cases} \tag{14}$$

It's immediately seen that the consensus time of voter model on a graph is essentially different from that (Equation 6) on a lattice. Systems with different degree distributions behave distinctively, which demonstrates that the connection between people affects voting to a dramatic extent. Since the graph model reflects more a real society than a lattice model, it's reasonable to say that the result above is more reliable and more close to the real voting process.

A remaining question is what will happen if the graph is infinite. To answer the question, we should investigate the density of interface between different spins n_{AB} as a function of time. In the numerical experiment by Castellano *et al.* [9], n_{AB} is found to form a plateau after certain time of evolution, as shown in Figure 2. The rewiring number p stands for the probability of a node being connected to nodes besides its geographical nearest neighbors. $p = 0$ is equivalent to a lattice and $p = 1$ to a random graph.

It's also clear in the figure that the more nodes there are in the network, the longer the plateau duration will be. The plateau represents a metastable state in the voting process. As $N \to \infty$, the system will stay in the metastable state, where two opinions coexist, and consensus won't be realized. This is understandable, just like the scenario of $D > 2$ in infinite lattice voter model: in a large society where individuals have many accesses to information, the possibility of a complete agreement is destroyed.

4 Modifications

In the voter model discussed in previous two sections, many approximations have been made, which leads the model inaccurate. For example, it's assumed that each neighbor exerts the same amount of peer pressure, which apparently fails to mirror the friendship hierarchy. To polish the model for a better description of the real voting process, some modifications have been made. One of them is to change the spin flipping probability Equation (1) in lattice voter model, which accounts for the unequal influence from different neighbors. Also, an external magnetic field can be applied so that every spin feels a global influence, which materializes the media propaganda in society. And the centrists, whom both the leftists and rightists try to win over, can be introduced [10]. Another modification can be adding "zealot(s)" [11], who will never change mind, in the system. Here, I present the work by Mobilia [11] on "zealot" in lattice voter model.

According to the analytical work by Mobilia [11], in infinite 1D and 2D lattice, the zealot will influence all the individuals in the system and leads

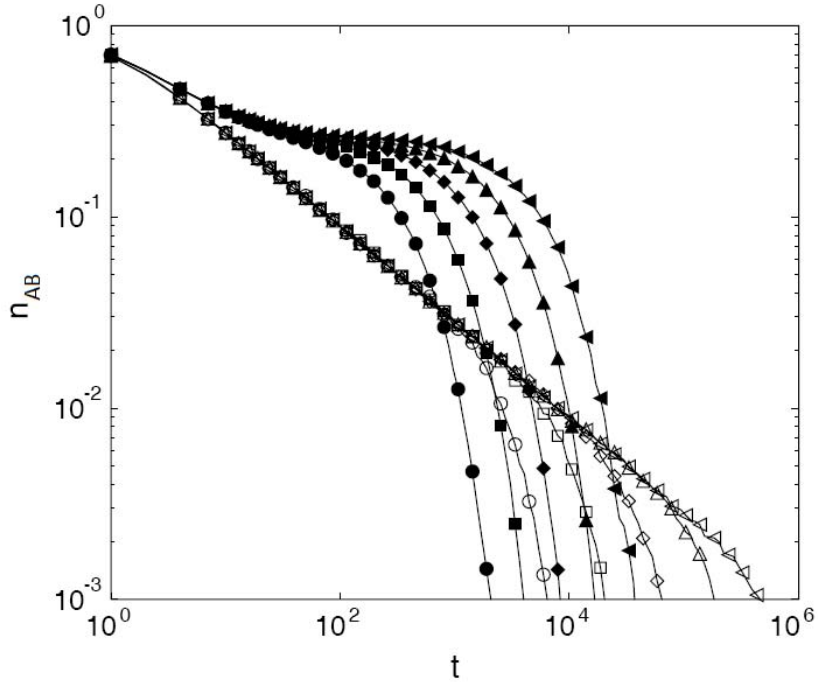

Figure 2: log-log plot of the interface density n_{AB} (normalized with respect to the initial value) versus time. Values are averaged over 1000 runs. Time is measured in Monte Carlo steps per site. Empty symbols are for 1D lattice case ($p = 0$). Filled symbols are for rewiring probability $p = 0.05$. Data are for $N = 200$ (circles), $N = 400$ (squares), $N = 800$ (diamonds), $N = 1600$ (triangles up) and $N = 3200$ (triangles left). (Figure reproduced from [9])

to a consensus in agreement with its own opinion. In higher dimensions, consensus cannot be achieved for infinite lattice, while individuals around the zealot finally hold the same opinion as the zealot does.

5 Conclusion

Voting is a process of extracting order out of chaos. The task of voter model is to visualize this process. In this term paper, voter model is briefly reviewed from a physical perspective. Firstly, lattice voter model is introduced due to its simplification and straightforwardness. It's concluded that in 1D and 2D lattice, consensus is always achievable regardless of the lattice size; while in higher dimensions, only finite systems can reach consensus. The consensus time dependence on lattice size in different dimensions is derived. Secondly, to better mimic the real human society, voter model on graphs is discussed. The consensus time as a function of system size is different from that in lattice, and consensus can never be realized in infinite systems. The huge influence of network topology on the model manifests that the connection among individuals determines the evolution of voting process dramatically. Finally, some modifications to the model are presented.

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