# Emergence of traffic jams in high-density environments

**Bill Rose**
**12/19/2012**
**Physics 569: Emergent States of Matter**

"Phantom" traffic jams, those that have no apparent cause, can arise as an emergent phenomenon in many models of traffic flow. These jams emerge when the density of traffic is sufficiently high. This survey paper will describe the basic elements of traffic flow that are consistent across all models and will also analyze the differences in predicted states that occur depending on the chosen model, making comparisons to experimental data when possible.

## Introduction

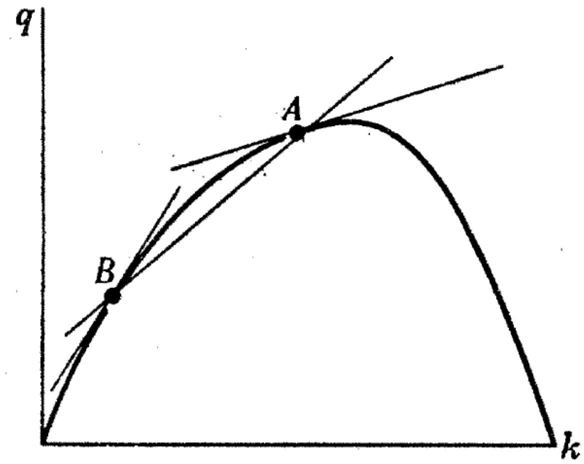
Due to the many human elements of traffic patterns, flow, and jams, one might expect it to be difficult to determine a simple set of equations to predict or model traffic. However, physicists and engineers have been using modeling techniques to study traffic patterns for several decades. There are two main categories of models. The first to be developed were macroscopic models which concern themselves only with the behavior of the entire system. The alternative, microscopic models, attempt to determine the behavior of individual cars and then to induce the behavior of the whole system from that of its constituent parts. Since models from both of these categories can be useful in different situations and for achieving different goals, this paper will discuss several models of each type.

An engineer's motivation for modeling traffic flow would likely be to learn something that would inform future highway or roadway design. A physicist might be more interested in the form of the equations involved, and how they relate to other physical phenomena and experiments. We will consider these models in regards to available experiment, and also with some interest in the engineer's perspective: that of applying the models to inform highway planning. This paper seeks to introduce several of these models, and provide analysis of their strengths and weaknesses. We hope to provide the reader with a broad look at the many types of models proposed for describing traffic flow, as well as to provide some history of how the field has developed over time.

## Fluid Dynamics Models

One of the earliest attempts to employ mathematical equations to study traffic flow was by M. J. Lighthill and G.B. Whitham, using the theory of kinematic waves that they developed. Their theory evokes kinematic waves as a means for transporting information about the state of flow of the system at a given location. These waves have velocity equal to the slope of the standard flow vs. concentration diagram for displaying highway throughput [1]. These kinematic waves applied to traffic are sent out when, for example, one car brakes, and then cars behind it react to the brake lights and also slow down, and then more cars respond to those brake lights [1]. Lighthill and Whitham note that their model is only useful on long, "crowded" roads, which they define as being in the regime where mean speed depends on the concentration of cars [1].

Lighthill and Whitham use their theory to describe shock waves in traffic, which occur when fast moving cars in an area of low density catch up to slow moving, densely packed cars: in other words, that moment when one hits the back of a traffic jam. If one knows the flow-density relationship for a given road or situation, one can model these shock waves graphically (Fig. 1b).



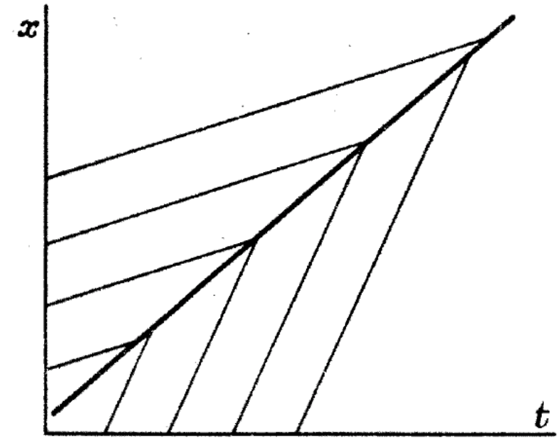



**Fig. 1a [1] (left):** Flow (*q*) as a function of concentration (*k*). This is a sample flow-concentration relationship. Lines tangent to the slope of points A and B represent the speed of kinematic waves, while the line between A and B shows the average speed of cars.

**Fig. 1b [1] (right):** Shock wave diagram showing position (*x*) as a function of time (*t*) corresponding to the flow-concentration diagram in Fig. 1a. The bold line shows the shock wave, and the other lines represent kinematic waves on either side of the shock wave, all of which intersect with it. In front of the shock wave (above the bold line), density is high, and kinematic waves move slowly, while behind the shock wave, density is low, and kinematic waves move quickly.

Using this model for shock waves, Lighthill and Whitham study how traffic jams propagate. Beginning with a model of a road that has an area of higher concentration where the shock wave will form, Lighthill and Whitham trace the movement and development of the shock wave. They find that it slowly propagates forward in time, and the shock wave produces a jam of cars behind it. Initially, the shock wave grows in strength, as more cars join the jam, but over time, it spreads out backwards until it eventually disappears. Lighthill and Whitham also apply their model to bottleneck situations, such as a lane closure or a tunnel, but we will keep this paper focused on free travel situations.

Lighthill and Whitham's shock wave model was an early attempt to gain a mathematical understanding of traffic jams, and as such, has as many flaws as strengths. In order to apply their model, one must already have either a theory or experimental data to provide a flow-concentration relationship. In addition, the functional relationship between flow and concentration may change depending on the time of day, the weather, or other external factors, which would make it very difficult to determine. Also, the model requires that one presume situations in which the concentration of cars, and therefore the mean-speed and the flow, vary at different points on the road.

M. R. Flynn et al. have a more specific theory using a macroscopic model that is designed to explain the shape of certain traffic jams. They begin with these standard fluid mechanics equations [2]:

$$\rho_t + (\rho u)_x = 0$$

$$u_t + u u_x + \frac{p_x}{\rho} = \frac{1}{\tau}(\tilde{u} - u)$$

In these equations, $\rho$ is the density of cars, $u$ is the speed, $\tilde{u}$ is the desired speed, $p$ is the traffic pressure, a quantity that increases with density, $\tau$ is a relaxation time, and subscripts represent differentiation with respect to that quantity, either time or position [2]. The behavior that the authors are trying to model was found in an experiment by Sugiyama et al. [3]. It is a traffic jam that moves backwards in position and forces cars to rapidly decelerate as they approach it, move very slowly for a brief period of time, and then accelerate freely out of it. One may experience this type of jam frequently while driving on busy freeways. To model this situation, the authors define the variables: $\eta = (x - st)/\tau$, and $c \equiv (d_p/d_\rho)^{1/2}$, where $s$ is the traveling wave speed, and $c$ is defined by the equation for the speed of sound in a compressible medium [2]. By introducing $\eta$ and $c$, we can combine our two initial fluid dynamics equations into one [2]:

$$\frac{d_u}{d_\eta} = \frac{(u-s)(u-\tilde{u})}{(u-s)^2 - c^2}$$

This equation can be solved by integration, provided that $d_u/d_\eta$ is finite at $u - s = c$ [2]. This means that $(\tilde{u} - u) = 0$ at this point, and this quantity goes to zero in such a way as to make the full equation finite. The waves that result are shown in Fig. 2. One can also look at the position of individual cars as a function of time and compare that to data collected from highway observations. Those relationships are shown in Fig. 3.

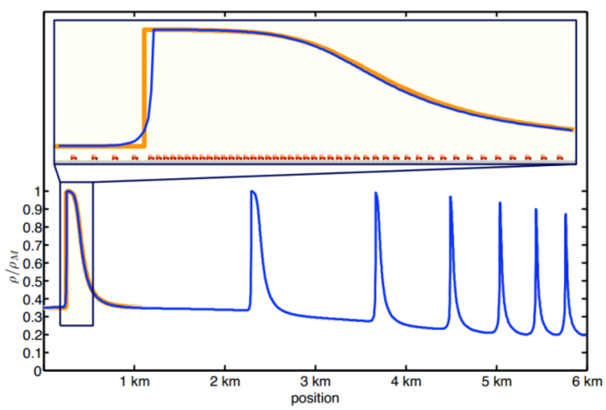



**Fig. 2 [2]:** Density waves as a function of position. $\rho_M$ is the maximum density. The orange line is the theoretical description outlined here, and the blue one is a numerical model obtained using a Lagrangian particle method, described in Appendix A of the referenced paper [2]. The top portion is a zoomed-in view of the first density wave.

One of the most interesting things to note about the data from Flynn et al. is that it shows a backwards traveling jam. This runs contrary to that proposed by Lighthill and Whitham, but seems to coincide with data taken in experiments and observations of real roadways. The authors also note that the functional form that they have derived for the density of cars matches that of detonation waves [2].

The Flynn et al. paper does an excellent job of providing a set of mathematical calculations to model one type of traffic jam that match up very well with experimental and observational data. While somewhat narrow in its scope, they have expanded upon this in other papers. They also have adjusted certain starting values to produce results that match experimental data, but have not provided a discussion of whether those initial values make sense on their own.

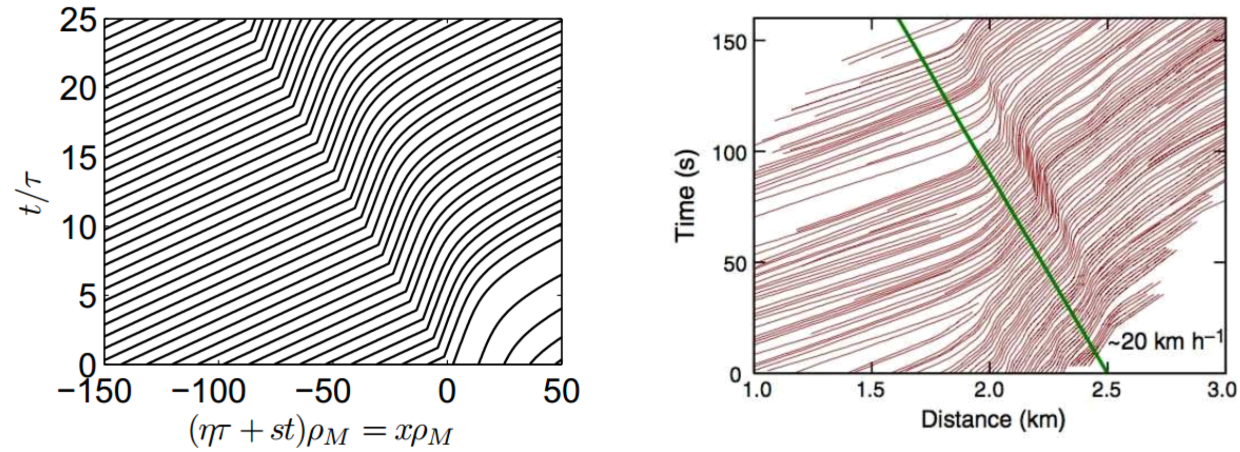



**Fig. 3a [2] (left):** Time vs. space plot of vehicle trajectories from Flynn et al.'s theoretical equations using $\rho_0/\rho_M = .35$, and $u\_0\ /\ \tilde{u} = .65$.
**Fig. 3b [3] (right):** Vehicle trajectories from aerial photograph by Treiterer and Myers. The green line shows the velocity of the jam cluster in Sugiyama's experiment.

## Gas-Kinetic (Boltzmann-like) Models

An early microscopic approach to modeling traffic flow was the gas-kinetic model. In this model, initially created by Prigogine and Herman [4], cars are modeled as particles in an interacting gas. The Boltzmann equation that governs the kinetic theory of gases is [5]:

$$\left(\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\nabla_r + \boldsymbol{F}\cdot\nabla_p\right) f(\boldsymbol{r},\boldsymbol{p},t) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$$

In this equation, $f(\boldsymbol{r},\boldsymbol{p},t)$ is the density of particles as a function of position, momentum, and time. $(\partial f/\partial t)_{\text{coll}}$ is the derivative of $f$ caused by collisions with other particles. $\boldsymbol{F}$ is an external force on the system. Prigogine et al. adapted this model to traffic by introducing a desired density function ($f_{\text{des}}(x,v)$) which the system wants to tend towards over a relaxation time $\tau_{\text{rel}}$ [5]. They then suggest that the analogue of the Boltzmann equation should take the form [5]:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial t}\right)_{\text{rel}} + \left(\frac{\partial f}{\partial t}\right)_{\text{int}}$$

Here, $(\partial f/\partial t)_{\text{int}}$ is the analogue of the collision term in the Boltzmann equation for gas particles, and $(\partial f/\partial t)_{\text{rel}}$ corresponds to the external force term $\boldsymbol{F}\cdot\nabla_p f(\boldsymbol{r},\boldsymbol{p},t)$ [5]. One simplification made by Prigogine et al. was that the whole system would relax at the same rate, so that $(\partial f/\partial t)_{\text{rel}} = -(f - f_{\text{des}})/\tau_{\text{rel}}$ [5]. A further assumption that made was that the desired speed distribution was the same at all points, regardless of the density at that point. These assumptions, and the model based on them, were shown by Paveri-Fontana to give unphysical results in the zero interaction, or low density, limit [6].

S. L. Paveri-Fontana provided some improvements on this Boltzmann-like theory of traffic. He introduces a series of assumptions about what happens when a fast moving car catches up to one moving more slowly on a multi-lane highway [6]:

1) *The "slowing down event" has prob. (1-P) and the passing event has prob. P, with* $0 \le P \le 1$. *If the fast car passes the slow one, its velocity is not affected at all.*
2) *The velocity of the slow car is unaffected by the interaction or by the fact of being passed.*
3) *Vehicle lengths can be neglected.*
4) *The "slowing down" process has instantaneous duration.*
5) *Only two-vehicle interactions are to be considered, multivehicle interactions being excluded.*
6) *The assumption of "vehicular chaos" is valid; namely,* $f_2(x', v', x, v, t) \cong f(x, v, t) f(x', v', t)$. *So that vehicles are not correlated.*

Some of these assumptions may seem to over-simplify actual highway driving, but they are necessary to reduce the equations to manageable form. These assumptions were presented by Paveri-Fontana, but also underlie the Prigogine methodology [6]. They lead to the equation [6]:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{int}} = f(x, v, t) \int_0^\infty dv'(1 - P)(v' - v) f(x, v', t)$$

Paveri-Fontana admits that these assumptions, and therefore this equation, are only valid in a dilute traffic limit, where the interaction is small.

The major change that Paveri-Fontana makes to the Prigogine model is to assign each driver a desired speed, rather than having an overall desired speed distribution. While a subtle difference, it makes a large difference in the system's behavior. Defining $f(x, v, t) = \int_0^\infty g(x, v, t, w) dw$, where *w* is the desired speed of a given vehicle, we can write the new Boltzmann-like equation [6]:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x}\right) g(x, v', t, w) + \frac{\partial}{\partial v}\left(\frac{w - v}{T} g(x, v', t, w)\right) \\ = f(x, v, t) \int_v^\infty (1 - P)(v' - v) g(x, v', t, w) dv' \\ - g(x, v, t, w) \int_0^v dv'(1 - P)(v' - v) f(x, v', t) \end{aligned}$$

This equation does not produce the same unphysical results in the zero interaction limit as the simpler Boltzmann-like equation given by Prigogine et al., and while it appears somewhat unwieldy, it is in fact only slightly more complicated than the Prigogine equation.

**Car-Following Models**

So far all the models we have looked at are adaptations of canonical physics equations – either fluid dynamics, or an interacting gas. Next, let us move onto models that specifically attempt to account for the vehicular nature of traffic. The first are a class of theories called car-following theories. The main supposition of these theories is that in dense environments, drivers will adjust their behavior based on the movement of the car in front of them. The simplest of these suggests that drivers will try to maintain a certain following distance behind the car in front of them for a given speed [7]. This suggests that if one car brakes, the car behind it, if already following at the desired distance, will brake also to maintain a safe cushion between cars. One slight modification to improve the realism of this model is to introduce a lag time of about a second between when one car accelerates and when the following car reacts.

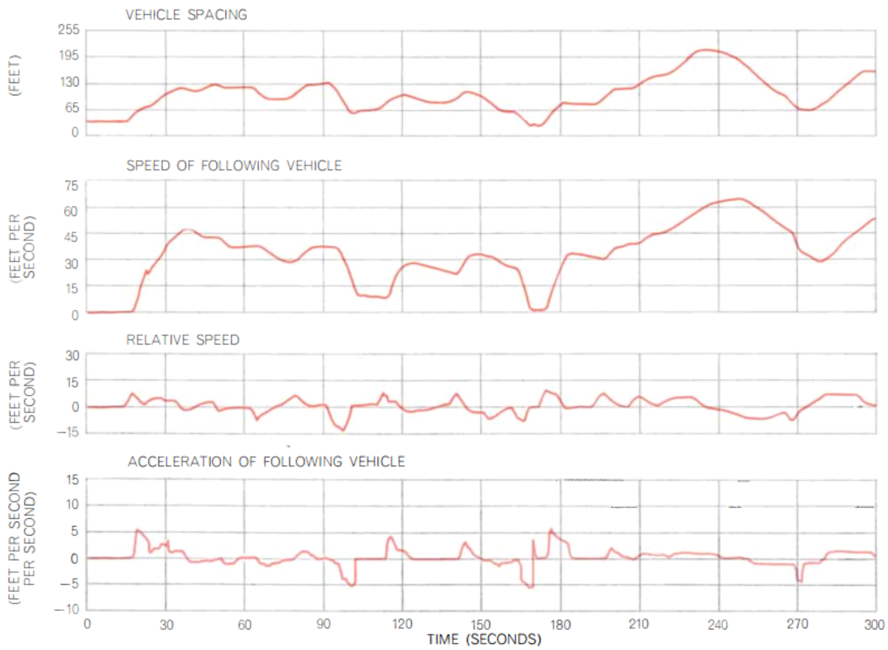



**Fig. 4 [7]:** Experimental data taken in a follow-the-leader setup where one car remains behind another. The data was taken using a spool of piano wire attaching the two cars with sensors on the spool to determine how far apart the cars were as a function of time.

However, experimental data (Fig. 4) suggests that an even more realistic model is that instead of trying to react primarily to the distance between one's car and the next, drivers primarily base their reactions on the difference in speeds between one's car and the next. Following some lag time, drivers will adjust their speed so that the distance between their car and the next car allows them sufficient time to reduce their speed to match that of the car in front of them [7]. The equation that Herman and Gardels present to explain this behavior is as follows [7]:

$$\frac{d^2x_{n+1}(t+T)}{dt^2} = \frac{G_{n+1}}{x_n(t) - x_{n+1}(t)}\left[\frac{dx_n(t)}{dt} - \frac{dx_{n+1}(t)}{dt}\right]$$

In this equation, *n* is a label for cars, with the $n + 1$ car being directly behind the $n^{\text{th}}$ car. *G* is a car's gain coefficient, and represents how strongly a driver reacts to the motion of the car in front. *T* is the reaction time of drivers, which experiments suggest is slightly less than one second. This equation is obviously valid only for the single-lane case, and Herman and Gardels refer back to a modified Boltzmann equation for modeling multi-lane flow, although one could also more directly modify the car-following theory to allow for multiple lanes. Other possible modifications to the model described here include introducing some desired velocity which cars will revert to when possible [5]. This would help extend the model to the dilute case where other cars are far away. Another extension would be to include stimuli from multiple cars in front, rather than just the one directly ahead [5]. Drivers-education teachers always stress that looking several cars down the road is important for avoiding accidents.

## Cellular Automata Models

Kai Nagel and Michael Schreckenberg developed a simple cellular automata model [8] that describes several important aspects of traffic flow, and that has been used as a basis for other more complicated models since then. Their model consists of a one-dimensional array of $L$ sites, with each site either having a one or a zero to represent being filled by a car or being empty. Secondly, each filled site has an integer velocity between zero and $v_{\max}$, which the authors chose to be five, for reasons to be explained later. From this starting point, they introduce four rules to govern motion of the cars. These are those four rules, as described by Nagel and Schreckenberg [8]:

1) **Acceleration:** *if the velocity v if a vehicle is lower than $v_{max}$ and if the distance to the next car ahead is larger than $v + 1$, the speed is advanced by one [$v \rightarrow v + 1$].*
2) **Slowing down (due to other cars):** *if a vehicle at site i sees the next vehicle at site $i + j$ (with $j \leq v$), it reduces its speed to $j - 1$ [$v \rightarrow j - 1$].*
3) **Randomization:** *with probability p, the velocity of each vehicle (if greater than zero) is decreased by one [$v \rightarrow v - 1$].*
4) **Car motion:** *each vehicle is advanced v sites.*

The model consists of many iterations of these four steps, applied to the entire one-dimensional array. One can see how these steps, while quite simple mathematically, resemble the standard motion of cars on a freeway. Step three, randomization, is necessary so that the model is not entirely deterministic, and is justified by fluctuations in the speeds of cars that are so common on freeways, as drivers lose attention for a few seconds, or take a sip of a drink, or answer a cellphone.

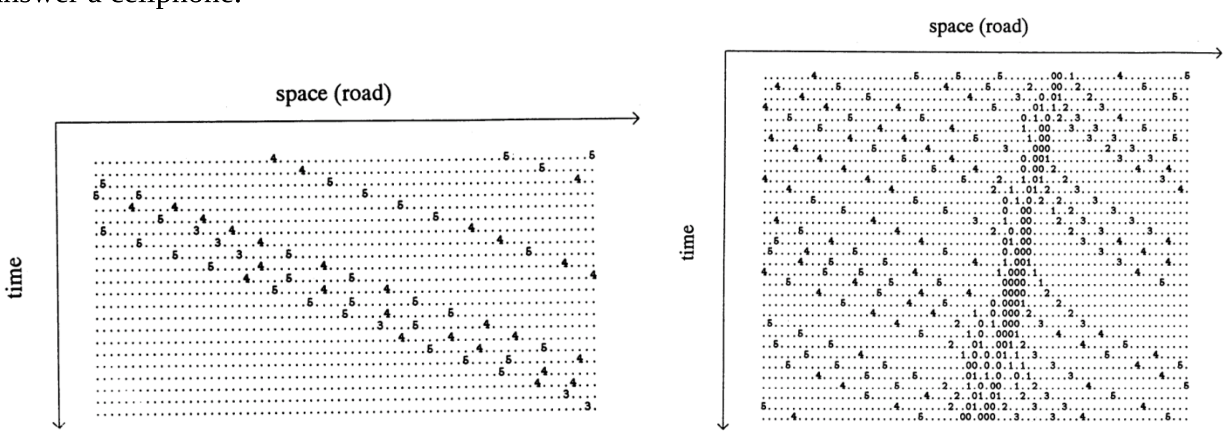
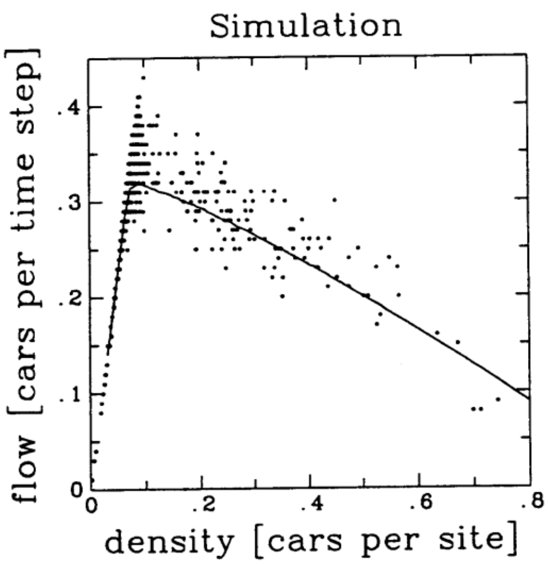



**Fig. 5a [8] (left):** A representative piece of Nagel and Schreckenberg's model with $\rho = 0.03$. Each number represents the velocity of a car at that location.
**Fig. 5b [8] (right):** Same as Fig. 5a, but with $\rho = 0.1$.

The authors define the density $\rho$ to be the number of times a certain site is occupied out of the total number of iterations as the number of iterations goes to infinity. The authors began the simulation with random placement of cars at the desired density $\rho$. They start all cars with zero velocity, and run the simulation until it reaches equilibrium ($10 * L$ iterations) before studying the results. As one expects from our real-world experiences, at low-densities, the authors find smooth, laminar flow (Fig. 5a). Cars may occasionally have to slow down just a bit for the car in front of them, but generally all cars are moving at speeds close to $v_{\max}$. However, at somewhat higher densities (Fig. 5b), the authors see mostly smooth flow punctuated by random clumps of cars moving very slowly, which are caused by the random deceleration of a car or cars.

For many highway engineers, throughput, or flow, is the quantity of highest interest, because this tells how many cars can be moved on the road in a given amount of time. Flow is simply the number of cars passing a certain point over a unit time, and is usually averaged over long time spans. Nagel and Schreckenberg performed their model for a range of different densities (Fig. 6a), and compared it to real data collected from highway measurements (Fig. 6b).



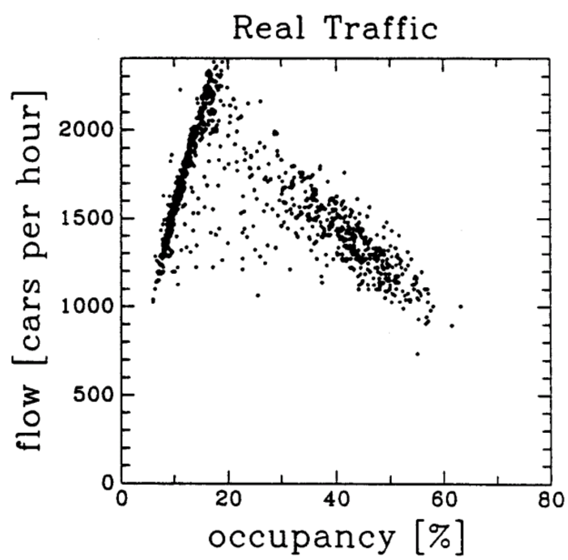




**Fig. 6a [8] (left):** Average flow of cars as a function of density from Nagel and Schreckenberg's model.
**Fig. 6b [8] (right):** Average flow of cars as a function of density collected from highway measurements.

## Coupled-map Lattice Models

Krauss, Wagner, and Gawron created a model that is similar to the cellular automata model of Nagel and Schreckenberg, but has a continuous spatial coordinate. As such, the equations they give for their model are [9]:

$$v_{\text{des}} = \min\left[v(t) + a_{\max}\, , v_{\max}\, , s_{\text{gap}}(t)\right]$$

$$v(t+1) = \max\left[0, v_{\text{des}} - \sigma n_{\text{ran},0,1}\right]$$

$$x(t+1) = x(t) + v(t+1)$$

Where $v_{\text{des}}$ is the free space to the next car, $a_{\max}$ is a car's maximum acceleration, $n_{\text{ran},0,1}$ is a random number between zero and one, and $\sigma$ is the maximum deceleration due to noise [9]. $\sigma n_{\text{ran},0,1}$ in this model is analogous to the probability p of reducing velocity by one unit in the Nagel-Schreckenberg model. Krauss, Wagner, and Gawron note that their model can be expressed as the continuous limit of a cellular automata model, assuming they set their parameters to certain values. The continuous model is especially different from the discrete one in the case where density is very high [9], because in this regime distances and velocities are small, so values between one and zero become very important. Accounting for these differences could also be achieved by decreasing the unit length scale of the discrete model, rather than going to the fully continuous limit.

Krauss, Wagner, and Gawron conclude their paper in part by providing descriptions of the different phases they find. My summaries of those phases are as follows, with physical analogues provided by Krauss, Wagner, and Gawron [9]:

$0 < \rho \leq \rho_{c_1}$: Free flow regime. In this density range, most cars move freely at or near their maximum velocity. Occasional clusters of a few cars occur, but quickly spread out again. The physical analogue is a dilute gas.

$\rho_{c_1} < \rho \leq \rho_{c_2}$: Two flow regime. For these densities, regions of free flow occur, but so do heavily jammed regions. The physical analogue is liquid in equilibrium with saturated vapor.

$\rho_{c_2} < \rho \leq 1$: Jammed regime. This state is one large traffic jam, with all cars moving slowly almost all the time. As seen in the free flow regime, occasional outliers exist, but these cars quickly become stuck again. The physical analogue to this state is a compressible liquid.

## Analysis and Conclusions

We have presented several of the most important types of traffic models that have been developed and used over the past fifty years. While some of the models described in this paper only deal with the behavior of traffic flow on one side of the transition or the other, all of them relate to the idea of an emergent behavior of the system, traffic jams, for sufficiently high densities of cars. They all attempt to describe the form of the system in different states, although the various models we discussed may contradict each other in certain cases.

The fluid dynamics models posit a relationship between density and flow, while the other models show that such a relationship exists from their initial assumptions. However, all models require some amount of fitting to external parameters to make them align with experiments. One important thing to test, which for the most part is not within the scope of this paper, is which theories can be calibrated for one set of observations or experiments, and then applied accurately to other highway situations. In other words, which of these models is actually most useful for informing future highway planning? Also of note is that some of these models only work in certain limits, like the Lighthill and Whitham model which applies only in high-density situations, the gas-kinetic models which apply only in the low-density regime, or the Krauss, Wagner, and Gawron model which is only substantially different from the cellular automata model in high-density environments. Additionally, only a few models take into account the possibility of multiple lanes of traffic. These are primarily the Boltzmann-like models, although the car-following models have been adapted for multiple lanes as well.

That the Boltzmann-like models account for multiple lanes makes them good candidates for applying their predictions to true highway situations. However, that they are only valid in the low-density regime negates much of their usefulness for studying traffic jams. The cellular-automata models, and the coupled-map lattice models based on them, can be fit to experiment and observations well, and seem to take more information into account in an accurate way than many of the other models discussed. Their one major weakness is that those mentioned here apply only to one lane. Perhaps a model which extended Nagel-Schreckenberg to multiple lanes by positing the behavior of drivers with cars next to them or when passing would be the best option for guiding roadway planning. Finally, note that other models may need to be considered if one is interested in studying traffic patterns in cities or areas with traffic lights, as these are altogether different from the highway models presented in this paper.

## References


[1] M. J. Lighthill and G. B. Whitham, *Royal Society of London Proceedings, Series A* **229** (1955) 317-345.

[2] M. R. Flynn, A. R. Kasimov, J.-C. Nave, R. R. Rosales, and B. Seibold, *arXiv* **math.AP** (2008) 0809.2828v2.

[3] Y. Sugiyama, M. Fukui, M. Kikuchi, K. Hasebe, A. Nakayama, K. Nishinari, S.-i. Tadaki, S. Yukawa, *New Journal of Physics* **10** (2008) 033001.

[4] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic*. New York: American Elsevier (1971).

[5] D. Chowdhury, L. Santen, and A. Schadschneider, *Physical Reports* **329** (2000) 199-329.

[6] S. L. Paveri-Fontana, *Transportation Research* **9** (1975) 225-235.

[7] R. Herman and K. Gardels, *Scientific American* **209** (1963) 35-43.

[8] K. Nagel and M. Schreckenberg, *Journal de Physique I France* **2** (1992) 2221-2229.

[9] S. Krauss, P. Wagner, and C. Gawron, *Physical Review E* **54** (1996) 3707-3712.