Phys 569 term paper

Symmetry Fractionalization and Topological Order

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December 2011

Abstract: In this article, I would first introduce a new phase of matter with symmetry fractionalization and symmetry protected topological orders. Such a system is beyond Landau's symmetry breaking theory. For a symmetry protected topological ordered state, even without symmetry breaking, it is still different from a conventional non-symmetry breaking one as it cannot be adiabatically connected to a trivial disordered state as long as the symmetry is protected. I would give the basic rules to classified different symmetry topological phases, based on the Matrix Product State Representation and the group cohomology of the symmetry. Then, a complete set of nonlocal string order would be defined in theses phases, which shows that such s symmetry fractionalization state is different from a trivial disordered state since its local entanglement is protected by symmetry and the irreducible entanglement in these phase of matter is responsible for its nontrivial topology.

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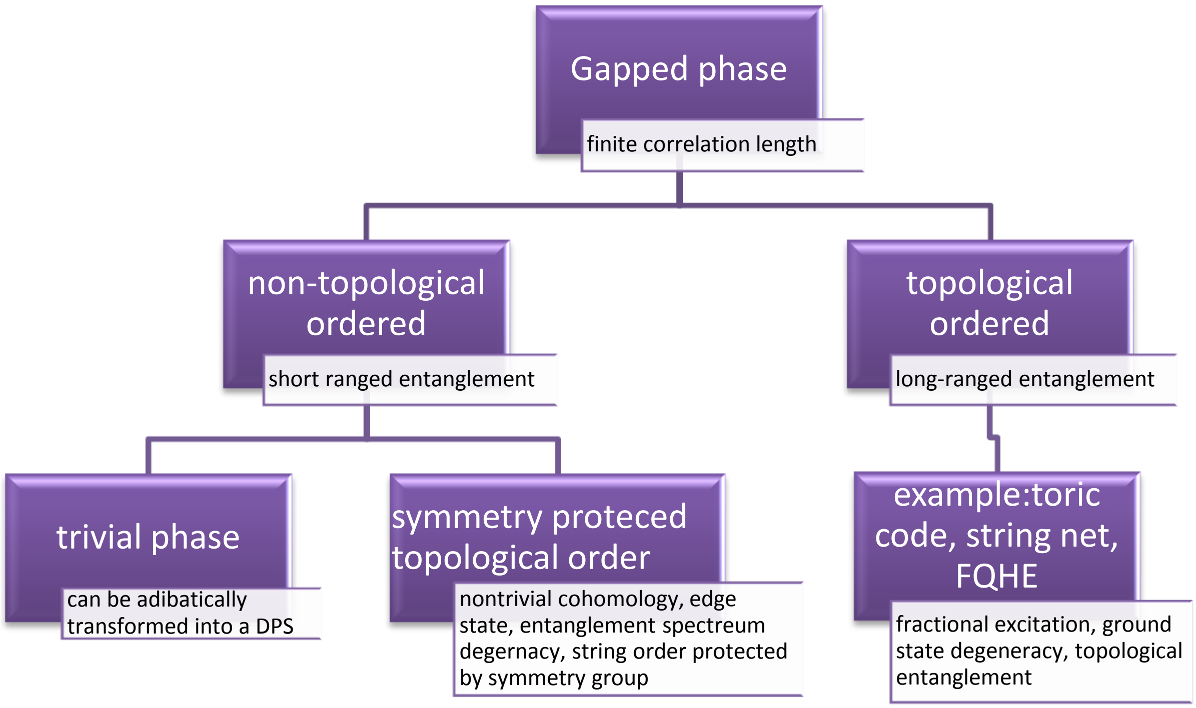
I. The breakdown of Ginzburg–Landau theory

For many years, the Ginzburg–Landau theory[1] for phase transitions had been regarded as the basic foundation of the phase transitions as well as the classification of different phases. Phases of matter are different to each other as they have different symmetry, e.g. breaking of spin rotation symmetry gives the magnetic order, breaking of translational symmetry gives the stripe phase, dimer solid etc.

However, the birth of fractional quantum hall effect[2], had lead this theory in to puzzle. Even two phases of matter shares the same symmetry, they cannot be adiabatically connected without a phase transition. Later on, there has been raised some exactly solvable models to show that they do exist such a "topological phase of matter", such as toric code model[3], string net model[4], quantum dimer model [5], whose ground states does not break any symmetry, but still distinguished from other trivial disordered states. What makes issue more interesting is that these system have ground state degeneracy depends on the topology, as well as abelian/non-abelian excitation and nonlocal order parameter.

In order to avoid the misunderstanding of some concept in the later content, here I would first give the definition of gapped topological states[6]. There are two equivalent definitions, which based on the Hamiltonian or the ground state.1) A topological phase of matter's Hamiltonian is gapped, and cannot be adiabatically connected to a trivial one without closing the gap. 2) A topological phase of matter is has a many-body ground state wave function which could not be transformed to a direct product state through a local unitary transformation. For such a system with topological order, it has ground state degeneracy, fractional excitation (abelian or non abelian), topological entanglement[7,8].

Apart from the "topological states of matter", we also have "symmetry protected topological (SPT) states[6]", these states do not have long-range entanglement, and can be adiabatically deformed into a trivial state. However, if I impose a symmetry constrain, such of phase of matter is distinguished from the trivial state. Therefore, symmetry protected topological state could be defined parallel as follows, 1) Its Hamiltonian is gapped, and cannot be adiabatically connected to a trivial one without closing the gap until you break a certain symmetry. 2) Its many-body ground state wave function could not be transformed to a direct product state through a local unitary transformation which preserves the symmetry. Thus, topological insulator and topological superconductor[9], upon this definition is a symmetry protected topological phase, which was protected by time reversal symmetry, PH symmetry, or else. In these symmetry protected phases, there usually exist edge states[10] (except parity symmetry protection), entanglement spectrum degeneracy and string order [11](not Ill defined sometimes).



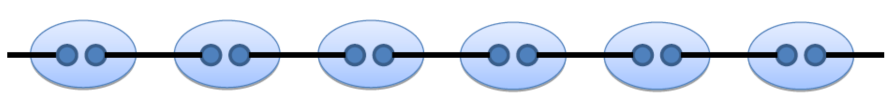
Draft 1. Diagram for the gapped phase classification

II. Classification of Symmetry protected topological phase

i. AKLT state----a toy model for fractional symmetry

Haldane conjecture[12] suggests that1-D Heisenberg model with integer spin should have a gapless excitation. In 1988, Affleck[13] et al. shown a spin 1 valence bond model which could characterize Haldane phase and its ground state is exactly solvable. The ground state, named AKLT state does not break any symmetry and have zero-mode edge excitation carrying spin ½.

The spirit of AKLT state could be caught by a "Projective Entanglement Pair state"(PEPS)[14]form.



Draft 2.AKLT state

Assume there is 2 spin ½ on each site, every spin ½ on the right side of the site is coupled to the left side spin ½ on nearest site, forming a singlet. Then, projecting the 2 spin ½ on each site into the totally spin 1 subspace would we get exactly the AKLT state. The AKLT state could be written via Schwinger-boson representation

$$|\varphi\rangle = \prod_i \left(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger\right)\left(a_{i+1}^\dagger b_{i+2}^\dagger - b_{i+1}^\dagger a_{i+2}^\dagger\right)\left(a_{i+2}^\dagger b_{i+3}^\dagger - b_{i+2}^\dagger a_{i+3}^\dagger\right) \dots\dots |VAC\rangle$$

Where $a_i^\dagger b_i^\dagger$ are Schwinger-bosons.

Here do we know that if chose the open boundary condition, we would get a free spinon on each edge, carrying spin ½. Thus, AKLT state would emerge a fractional "zero-mode edge state" carrying spin half-integer, although the system we are dealing with is a spin 1. Note that the AKLT state is a short ranged correlated state which does not break any symmetry, neither spin rotation symmetry or translational. However, it is still different from a trivial direct product state $|\psi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \dots\dots\dots\dots$ with the same symmetry which could be distinguished from their edge state, their projective representation, nonlocal string order[11] and entanglement spectrum. We cannot adiabatically connect such two states if we maintain a certain symmetry, e.g. $D_{2h,}$ time reversal, parity, etc. Thus, AKLT is the first example with fractional symmetry and its topological character is protected by symmetry. In order to give a clear and rigid statement for its nontrivial symmetry protected topology, I would introduce Matrix product state in the following content.

The development of Matrix product state[14] (MPS) and Tensor network states[15] (TPS) gives us an effective way to describe the many-body wave function. It had been shown that for any state with finite correlation length, we can use a finite dimension Matrix or Tensor to describe the wave function.

$|\Phi\rangle = Tr\,(\sum M_i^{zi}\, M_i^{zi} \dots\dots\dots\dots\dots\, M_i^{zi} |z_i \dots\dots\dots\dots\rangle)$

z_i is the local basis for i site. And M_i^{zi} is the matrix form the $z_{i.}$ For AKLT state, we can easily tell its MPS through its Schwinger-boson representation.

$$|\varphi\rangle = \prod_i \left(a_i^\dagger \;\; b_i^\dagger\right)\begin{pmatrix} b_{i+1}^\dagger \\ -a_{i+1}^\dagger \end{pmatrix}\left(a_{i+1}^\dagger \;\; b_{i+1}^\dagger\right)\begin{pmatrix} b_{i+2}^\dagger \\ -a_{i+2}^\dagger \end{pmatrix}\left(a_{i+2}^\dagger \;\; b_{i+2}^\dagger\right)\begin{pmatrix} b_{i+3}^\dagger \\ -a_{i+3}^\dagger \end{pmatrix} \dots\dots |VAC\rangle$$

$$= |\varphi\rangle = \prod_i \begin{pmatrix} a_i^\dagger b_i^\dagger & b_i^\dagger b_i^\dagger \\ -a_i^\dagger a_i^\dagger & -a_i^\dagger b_i^\dagger \end{pmatrix} \dots\dots |VAC\rangle$$

$$= \prod_i \begin{pmatrix} |0\rangle & \sqrt{2}|-1\rangle \\ -\sqrt{2}|1\rangle & -|0\rangle \end{pmatrix} \dots\dots |VAC\rangle$$

If I choose the time reversal basis, $|x\rangle = \frac{1}{\sqrt{2}}|-1\rangle - |1\rangle$, $|y\rangle = \frac{i}{\sqrt{2}}|-1\rangle + |1\rangle$, $|z\rangle = |0\rangle$. I get, $M_X = \sigma_{x,}$ $M_Y = \sigma_{y,}$ $M_Z = \sigma_{z,}$ such a formalism is invariant under D_2= {exp iπ$S_{X,}$ exp iπS_Y, exp iπ$S_{Z,}$ I} group. Thus, for each element g in $D_{2,}$ as the AKLT state is invariant under $D_{2,}$ u(g) $|\varphi\rangle$= $\exp(i\eta)|\varphi\rangle$, the MPS matrix should also change as $M'^{zi}_i = e^{i\theta(g1)}\, U(g)^\dagger\, M_i^{zi}\, e^{i\beta}\, U(g)$[6], for the AKLT state, I have $U(\exp i\pi Sx) = i\sigma_{x,} U(\exp i\pi Sy) = i\sigma_{y,} U(\exp i\pi Sz) = i\sigma_{z,}$ as we know that the group element in D_2 $\exp i\pi S_X * \exp i\pi S_Y = \exp i\pi S_{Z.}$ However, for U(g), we find $U(\exp i\pi Sz) = -U(\exp i\pi Sx) * U(\exp i\pi Sy)$, Thus, U(g) forms a projective representation of group $D_{2,}$ which satisfied $U(g_1 g_2)= \exp i(\theta_{12}) U(g_1)^* U(g_2)$. This extra phase in projective representation is extremely important for AKLT's nontriviality. For a trivial direct product state $|\psi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots \ldots \ldots \ldots$, the U(g) forms the linear representation instead.

The parent Hamiltonian of AKLT state could be written via projective operator[13].

$$H=\sum_{<i,j>} \frac{1}{3} (S_i \cdot S_j)^2 + (S_i \cdot S_j),$$

The $\frac{1}{3}(S_i \cdot S_j)^2 + (S_i \cdot S_j)$ term projects the nearest 2 spin 1 into the S_{total}=0 or 1 subspace, thus, there must form a singlet to ensure this projection, which gives the AKLT state. For open boundary condition, the AKLT state have a dangling spin ½ on each end, thus gives a zero spinon mode and 4-fold degeneracy. The AKLT state have a short ranged correlation function C(i,j)~exp- (i-j)/3, and preserves the translational symmetry. However, there is a hidden AF order in AKLT state, and thus a non-local string order could characterize it.

If we expanded the AKLT state in the on site S_z basis, the configuration of each term would be like follows,

100000-1001-1001000-10001-1000010000-1010-1001-100000,

Seems to be disordered, however, if we eliminate all the 0 terms in these configurations,

1[redacted]-1[redacted]1-1[redacted]1[redacted]-1[redacted]1-1[redacted]1[redacted]-1[redacted]1[redacted]-1[redacted]1-1[redacted],

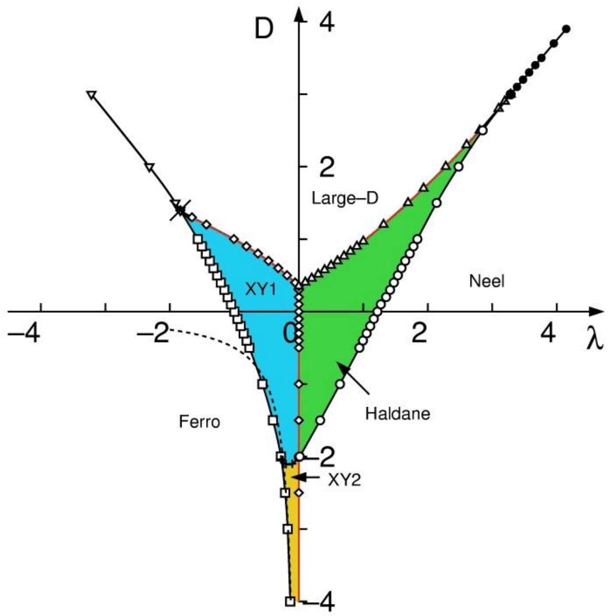
You would magically find it has a hidden dilute AF order[11]. To characterize this hidden order, we can define a string order parameter.

$$\langle \psi_{AKLT} | S_{n-1}^z \exp(i\pi \sum_n^m S_i^z) S_{m+1}^z | \psi_{AKLT} \rangle = 4/9$$

This string order does not only exist in the AKLT state, in fact, in a wide region of Haldane phase have such order parameter to be nonzero. The Heisenberg model with uniaxial anisotropy with D_2 symmetry,

$$H = \sum_{<ij>} S_i^x \cdot S_j^x + S_i^y \cdot S_j^y + \lambda S_i^z \cdot S_j^z + D(S_i^z)^2$$

In the whole Haldane phase, there exist a string order $\langle \psi_G | S_{n-1}^z \exp(i\pi \sum_n^m S_i^z) S_{m+1}^z | \psi_G \rangle$. In the later discussion, I would see that the existence of the string order parameter id related with the irreducible entanglement protected by the symmetry. However, due to the non-uniqueness of the fixed point upon tensor network renormalization[16], the definition of string order should be modified.



Draft 3. Phase diagram of the Heisenberg model with uniaxial anisotropy from W. Chen et al. PRB 67, 104401 (2003)

The string order parameter could be transformed into the correlation function if we make a nonlocal duality transformation to map the Hamiltonian to the dual space[17].

The original Heisenberg model with uniaxial anisotropy,

$$\mathrm{H} = \sum_{<ij>} J(S_i^+ \cdot S_i^- + h.c) + VS_i^z S_i^z + \frac{U}{2}(S_i^z)^2$$

If I do a non-local unitary transformation which preserves the D_2 symmetry,

$$U = \sum_{j<k} \exp(S_j^z S_k^x)$$

The Dual-Hamiltonian reads

$$H^{dual} = -J\sum_{<ij>} S_i^x S_j^x - S_i^y(\exp(i\pi S_i^z + i\pi S_j^x))S_j^y - VS_i^z S_i^z + \frac{U}{2}(S_i^z)^2$$

The string order parameter now becomes the correlation function

$$\langle S_{n-1}^z \exp(i\pi \sum_n^m S_i^z) S_{m+1}^z \rangle \rightarrow \langle S_{n-1}^z S_{m+1}^z \rangle$$

The new Hamiltonian in the dual space H^{dual} is still gapped with a finite correlation length and have a D_2(Z2*Z2) symmetry. Thus, the non-vanishing correlation function reveals a spontaneous symmetry breaking of the ground state. Note that in the Haldane phase of the original Hamiltonian have 4 folded degeneracy, due to the zero mode edge state. In the dual Hamiltonian after the nonlocal unitary transformation, the "zero mode edge state" flows into the bulk and gives the 4 states which breaks the discrete Z2*Z2 symmetry. In sum, through such a duality transformation, we transform a symmetry protected topological phase into a symmetry breaking one. Such s duality is very universal in one dimensional system. The very famous example is the one dimensional kitaev chain[18], which exhibit a free majorana mode on the edge, could also dual to a Z2 symmetry breaking phase. The main idea here is to change the degenerate edge state into the bulk and thus becomes a symmetry breaking phase. However, such approach usually fails in two dimensions as it is almost impossible to annihilate the gauge even the transformation is nonlocal.

ii. Classification of symmetry protected topological phases based on cohomology group

In the former content, I had introduced MPS representation for all gapped system. An advantage for MPS is that as long as we had write down a wave function in MPS, it is very easy to find its parent Hamiltonian whose GS is exactly the MPS state. Thus, instead of classified all gapped phases through the Hamiltonian, which has less chance to be exactly solvable, we can first write out a wave function in MPS, and classified the MPS.

For a symmetry group G, if a physical state Φ is invariant under G, for any g_1 (an element in group G), $g_1 |\Phi\rangle = e^{i\alpha} |\Phi\rangle$. Meanwhile, The matrix should change in the following style under a symmetry operator g_1, $M'^{zi}_i = e^{i\theta(g1)} U_1^{\dagger} M_i^{zi} U_1$. Here $e^{i\theta(g1)}$ form the 1-d representation of the symmetry group, and U_1 form the projective representation of the symmetry group[6]. The phase factor $e^{i\theta(g1)}$ is related with translational symmetry, so when translational symmetry is absent, we can always change the basis with a phase twist to adjust the $e^{i\theta(g1)}$ into the matrix.

For the group element $g_{12} = g_1 * g_2$, U changes as $U_{12} = e^{i\Phi 12} U_1 * U_2$, two elements times together would different from the element U_{12} up to a phase, that is projective representation. The symmetry operator representation acts on the state forms a linear representation, while symmetry operator representation on the matrix forms a projective representation.

We would realized that here U_1 is not unique, we can add a phase to U_1, $e^{i\beta} U_1$ also follows the relation that, $M'^{zi}_i = e^{i\theta(g1)} (e^{i\beta} U_1)^{\dagger} M_i^{zi} e^{i\beta} U_1$. Thus, the phase factor $e^{i\Phi 12}$ in "$U_{12} = e^{i\Phi 12} U_1 * U_2$" is not unique, since we can apply an arbitrary phase to U, thus we would define the equivalent class of $e^{i\Phi 12}$, where $e^{i\Phi 12} \sim (e^{i\beta 2} e^{i\beta 1} / e^{i\beta 12}) e^{i\Phi 12}$. For different cyclotron, $e^{i\Phi}$ cannot transform to each other, thus $e^{i\Phi}$ belongs to different equivalent class if they cannot connect through a $(ei^{\beta 2} ei^{\beta 1} / ei^{\beta 12})$ phase[6].

As I have defined the equivalent class here,we can classified different state with the same symmetry G with their Matrix product state, or more precisely, equivalent class of the $e^{i\Phi}$, each of this class is a group member of the 2nd cohomology group H(g, U(1))[6]. Note that if I impose on translational symmetry, then different $e^{i\theta(g1)}$ would also classified different states.

Below is a table of the classification of symmetry protected phase in one dimension.

Symmetry of Hamiltonian	Number of Different Phases
None	1
$SO(3)$	2
D_2	2
T	2
$SO(3)+T$	4
D_2+T	16
Trans. $+U(1)$	∞
Trans. $+SO(3)$	2
Trans. $+D_2$	$4\times 2=8$
Trans. $+P$	4
Trans. $+T$	2
Trans. $+P+T$	8
Trans. $+SO(3)+P$	8
Trans. $+D_2+P$	128
Trans. $+SO(3)+T$	4
Trans. $+D_2+T$	64
Trans. $+SO(3)+P+T$	16
Trans. $+D_2+P+T$	1024

Draft 4 classification of SPT phase in 1D, from Xie Chen et al, arXiv:1103.3323

Symmetry protected topological states classification shows that such phases of matter have nontrivial cohomology in their wave function. However, regardless of the completeness of such classification, how to perceive the SPT order in a more physical and direct way?

Below I would illustrate two ways to suggest the physical picture of SPT state—edge state active operator and string order. These physical parameters including the cohomology all comes from the irreducible entanglement protected by symmetry.

From the example of AKLT state, we can clearly realized that with the same SO(3) symmetry, AKLT state have an edge of half integer spin while trivial DPS state have integer spin. Thus, in the PEPS language, the 2 virtual spins on each site is integral or half integral before projection would definitely give gapped state with different topology. Since spin half integer have projective representation under SO(3) and integer spin is linear representation. Thus, edge state is an effective way to distinguished different SPT phase. However, only spin integer or half integer on the edge is not enough to tell all phases from each other. For example, for a spin chain with onsite D_2 symmetry and time reversal symmetry, we can write 4 distinguished wave functions, with the same symmetry but different cohomology under D_{2h}.

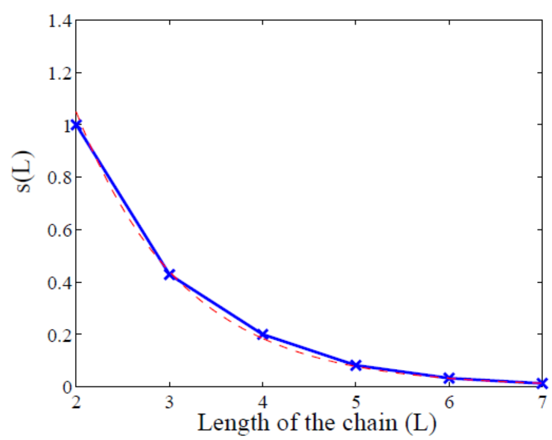
phase	M_X	M_Y	M_Z	H ($\gamma=\frac{1}{2(a^4+b^4+c^4)}$, $h_0=-\sum_{<ij>}c^4\gamma\left(S_{xi}^2S_{yj}^2+S_{yi}^2S_{xj}^2\right)+b^4\gamma\left(S_{xi}^2S_{zj}^2+S_{zi}^2S_{xj}^2\right)+a^4\gamma(S_{yi}^2S_{zj}^2+S_{zi}^2S_{yj}^2)$)
S_0	$a\sigma_x$	$b\sigma_y$	$c\sigma_z$	$\sum_{<ij>}\left(\frac{1}{4}+b^2c^2\gamma\right)S_{xi}S_{xj}+\left(\frac{1}{4}+a^2c^2\gamma\right)S_{yi}S_{yj}+\left(\frac{1}{4}+\right.$

				$a^2b^2\gamma\Big) S_{zi}S_{zj} + \left(\frac{1}{4} - b^2c^2\gamma\right) S_{yzi}S_{yzj} + \left(\frac{1}{4} - a^2c^2\gamma\right) S_{xzi}S_{xzj} + \left(\frac{1}{4} - a^2b^2\gamma\right) S_{xyi}S_{xyj} + h_0$
S_X	$ia\sigma_x$	$b\sigma_y$	$c\sigma_z$	$\sum_{<ij>}\left(\frac{1}{4} + b^2c^2\gamma\right) S_{xi}S_{xj} + \left(\frac{1}{4} - a^2c^2\gamma\right) S_{yi}S_{yj} + \left(\frac{1}{4} - a^2b^2\gamma\right) S_{zi}S_{zj} + \left(\frac{1}{4} - b^2c^2\gamma\right) S_{yzi}S_{yzj} + \left(\frac{1}{4} + a^2c^2\gamma\right) S_{xzi}S_{xzj} + \left(\frac{1}{4} + a^2b^2\gamma\right) S_{xyi}S_{xyj} + h_0$
S_Y	$a\sigma_x$	$ib\sigma_y$	$c\sigma_z$	$\sum_{<ij>}\left(\frac{1}{4} - b^2c^2\gamma\right) S_{xi}S_{xj} + \left(\frac{1}{4} + a^2c^2\gamma\right) S_{yi}S_{yj} + \left(\frac{1}{4} - a^2b^2\gamma\right) S_{zi}S_{zj} + \left(\frac{1}{4} + b^2c^2\gamma\right) S_{yzi}S_{yzj} + \left(\frac{1}{4} - a^2c^2\gamma\right) S_{xzi}S_{xzj} + \left(\frac{1}{4} + a^2b^2\gamma\right) S_{xyi}S_{xyj} + h_0$
S_Z	$a\sigma_x$	$b\sigma_y$	$ic\sigma_z$	$\sum_{<ij>}\left(\frac{1}{4} - b^2c^2\gamma\right) S_{xi}S_{xj} + \left(\frac{1}{4} + a^2c^2\gamma\right) S_{yi}S_{yj} + \left(\frac{1}{4} + a^2b^2\gamma\right) S_{zi}S_{zj} + \left(\frac{1}{4} + b^2c^2\gamma\right) S_{yzi}S_{yzj} + \left(\frac{1}{4} + a^2c^2\gamma\right) S_{xzi}S_{xzj} + \left(\frac{1}{4} - a^2b^2\gamma\right) S_{xyi}S_{xyj} + h_0$

Table1. Parent Hamiltonian and MPS of 4 SPT phase with D2h symmetry, from Z-X Liu et al. Phys. Rev. B 84, 075135 (2011)

From above table, we can see that the four gapped phases are distinguished from each other under D_{2h} protection, but with the same spin ½ on the edge (as the irreducible dimension of the matrix is 2, which contains the degree of freedom as a spin 1/2). Thus, we need more information about the edge to distinguish them.

Since the edge carries a free spin ½, if we apply a magnetic field in different polarized direction. As it breaks the time reversal symmetry, the edge states would be lifted if we add a Zeeman term in the Hamiltonian. For AKLT case, it is obvious that both Zeeman field in x, y, z direction would lift degeneracy. However, in $S_x/S_y/S_x$ phases, the edge spin is anisotropic and only field applied in x/ y /z direction can lift up the GS[10].



Draft 5. Numerical result showing that in Sx phase, if applying Zeeman field in y and z direction, the energy gap lifted between edge states would disappear in the thermal dynamic limit, from Z-X Liu et al. Phys. Rev. B 84, 075135 (2011)

[Type text]

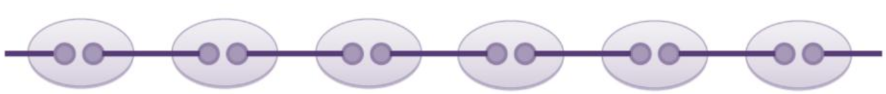
The above is just a simple case, to generalize it to a more universal case, we have to define a series of edge active operator $\{O_i\}$. In the physical space, the O_i transforms under symmetry group element as $u^{\dagger}(g)O_i u(g) = \eta O_i$. Meanwhile, in the MPS space, the pauli matrix (here the MPS dimension is 2, so pauli matrix is enough to expand the whole space, if the MPS dimension is higher, we can use other Gamma matrix, the process is the same) $U^{\dagger}(g)\sigma_i U(g) = \eta\sigma_i$ (here the U(g)is the transformation for the matrix of MPS)[10]. If O under symmetry operation have the same set of value η with one pauli matrix under all group element g, then we can say that the physical operator acts in the real system have the same algebra of pauli matrix in the MPS state, thus it should behavior as the edge active operator which lifts the edge degeneracy.

Likewise, we can then define the string order parameter $O_{n-1}^{i}\exp(i\pi\sum_{n}^{m} O_i^{i})O_{m+1}^{i}$. Here O_i corresponds to the edge active operator I find I before.

III. Entanglement spectrum and topology

Till now, I have defined edge active operator and string order parameter to characterize the different gapped phases. However, what makes a phase topological (under symmetry protection)? For a conventional symmetry breaking phase, we know that is the breaking symmetry and local order tells each states of matter from each other. Then what about topological states? From the previous section, I have generalized that string order and edge state could tell different phases of matter apart. But what is the driving force of such phase of matter to present "nontrivial topology"? Topological states as well as symmetry protected topological states of matter only exist in quantum system. Then it is natural the concept of symmetry breaking and local order, both appear in classical and quantum system, cannot classified also states of matter. Thus, what I need is a physical property unique in quantum. That is, quantum entanglement.

The entanglement entropy had already played an important role in quantum phase identifications. For critical system, the entanglement entropy scaling factor c is just the central charge of the CFT[19]. In gapped systems, if the system does not have topological order, the entanglement entropy should obey area law, otherwise, there would be an extra part from the area law which tell us the anyon excitation in the bulk[7,8]. For a symmetry protected topological phase, there should be no long ranged entanglement, thus no topological entropy. However, even it obeys the area law, the entanglement spectrum is still physically different from the trivial state. The spectrum is always degenerate or gapless. In fact, the SPT phase could have a nontrivial "topological character" since the some part of the entanglement is irreducible under symmetry protection[20]. In one dimension, this character is easy to verify through the wave function fixed point under tensor network renormalization[16].



Draft 5. fixed point state

Above graph is a picture for the fixed point wave function for tensor network RG. On each site, there are 2 virtual spins S on each site, every spin is bonded with a maximum entangled state with the other site while two spin s on the same site have no entanglement. Thus, if there is a symmetry constrain, we cannot make a local unitary transformation which preserves the symmetry to disentangle this state. For instance, if the virtual spin on these site are half-integral, no matter what local unitary transformation we add to the state, the state cannot be transformed to a direct product state $|\psi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \dots\dots\dots\dots$, as this state have different eigenvalue with the fixed point state(half integer) under time reversal operator, and the local unitary transformation preserves the symmetry. In this way, the local entanglement should always exist under symmetry protection and the entanglement spectrum should always have 2 folded GS degeneracy.

We can get a more general argument in all 1D systems, in one dimension fixed point states, the virtual spin S on each site is related with the irreducible dimension of the projective representation. Thus, if the for the group element in the cohomology group, the irreducible dimension directly suggests the entanglement spectrum degeneracy. Thus, for any projective representation, the dimension is at least 2, thus the entanglement spectrum have degeneracy protected by symmetry. Thus, we can do nothing to disentangle the state until the symmetry is damaged or there appears a phase transition which damages the GS manifold.

In 2D, the symmetry protected topological order and entanglement spectrum is still controversial. Numerical result had shown that the entanglement spectrum in a SPT phase is gapless and similar with the edge spectrum[21]. There is also analytical analysis showing that for free fermion models or interacting fermion with chiral edge state, the entanglement spectrum and edge spectrum should have one to one correspondence[22,23]. In FQHE system, it was shown that if we do the cut in the particle number space instead of real space, the entanglement spectrum would be similar to the bulk energy spectrum, and the braiding of the anyon quasiparticle[24]. In sum, entanglement is an efficient way to probe the topology of the system and through entanglement spectrum, we can get most information (excitation, braiding) of the system from the GS manifold.

IV. Reference

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