Fractional excitation in low dimensional system

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Abstract

The origin of bosons and fermions is a fundamental problem in physics. However, in some low dimensional condensed matter system, the bosons and fermions can be an emergent quasi-particle in some interacting system. The interaction here can changes the exclusion statistics of the particles in the original system. Another interesting emergent phenomenon is the fractionalization in these low dimensional system which is composed by particles with integer charges. I will discuss these emergent phenomenon in fractional hall liquid and some one dimensional systems in a unified picture.

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Physicists are like Alchemists. With some basic principles and materials they can create some new materials and even a whole universe. These phenomena are emergent, in the sense that they may be totally different and unrelated with the original system. In Fig.(1), there are four photos for four different emergent phenomena. Fig.1(a) is a trivial example. There are some bright spots on the black background. These bright spot can possibly be generated by different light sources: light bulks, matches or even the distant stars. We cannot distinguish which is the correct light sources until we come close to them and find out that they are fireflies.

We know that water may be the most popular material on the world. You can find it everywhere. However, it can be very dangerous some times. The second photo in Fig.1 shows the shock wave on the sea coast. Different from the sound wave, the shock wave is explained by a non-linear transport equation.

$$u_t + uu_x = 0 \tag{1}$$

The energy stored in the shock wave is really powerfully and can be used for surfing. It is fun and dangerous, I once jumped into the shock wave in Fig.1(b) when it hit the sea coast and totally lost my mind for three seconds. The third photo is the ripples on the icicle. At first, the surface of the icicle is smooth, a small perturbation in the flow of water on the surface will create ripple. In nature, the wavelength for the ripple is determined by the heat transport properties of the water flowing on the ice and roughly a constant which is about $1cm$. The ripple on the icicle moves on the reverse direction of the growth of icicle and the wavelength can be tuned by changing the water supply which has been proved in some experiments[1]. The two phenomena in Fig.2(b) and (c) belong to the non-equilibrium systems and are still open questions now. The following discussion will focus on some equilibrium systems, which are much simple to deal with.

The last photo in Fig.(1) is the star trail where the stars rotating around the north celestial pole and forms some closed loops. These closed loops form a pattern on a two dimensional manifold and have the same structure as the lowest landau level in a symmetric gauge. We will discuss the related physics in this paper. However, before discussing this two dimensional emergent phenomenon, let's first review some one dimensional interacting systems.

Consider a model with N bosons on a ring of circumference L. If there is no interaction between the bosons, the ground state will condensate to a $k_i = 0$ state, where k_i is the momentum for each particle. We then can add short-range repulsive interaction between the bosons. For simplicity, we consider zero-point interaction potential, the Hamiltonian looks like this

$$H = \int [-\Psi^\dagger(x)\partial_x^2\Psi(x) + g\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)] \tag{2}$$

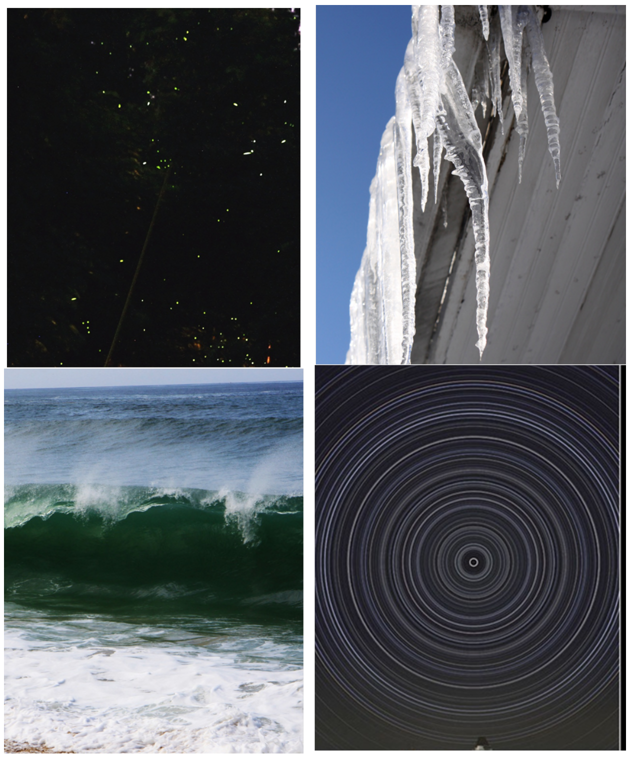

Figure 1: (a) Fireflies at Crystal lake park at Urbana in the summer of this year. (b) Shock wave at Waimea Bay of Oahu island in the last winter. (c) Icicle on the roof over the backyard entrance, taken at last winter. (d) Star trail Over Vienna. This photo is taken from this website:http://homepage.univie.ac.at/peter.wienerroither/indexe.htm

The equation of motion for this Hamiltonian is

$$i\partial_t\Psi = -\partial_x^2\Psi + 2g\Psi^\dagger\Psi\Psi \tag{3}$$

This model is called Lieb-Liniger model[2]. In the first-quantized language, the Hamiltonian has this form:

$$H = \sum_i P_i^2 + g\sum_{i<j}\delta(x_i - x_j) \tag{4}$$

We make the ansatz that the wavefunction can be written like this:

$$\Psi(x_1, ...x_N) = \sum_P A_p e^{\sum_n ik_{P(n)}x_n} \tag{5}$$

for the order $x_1 < x_2 < ... < x_N$, where the P represents all the possible permutation of the quasi-momentum k. The coefficients $A(P)$ and $A(P')$ are related by a pure phase shift in the permutation process, which can be decomposed into a product of two particle collision on a line. Notice that in 1d, after the two particle collision process, the momentums can only remain the same or exchange with each other due to the dimensional restriction. Thus the permutation and collision has the same physical meaning. For instance, if we consider the collision between particle 1 and 2, the phase shift is:

$$A(P) = \frac{k_1 - k_2 + ig}{k_1 - k_2 - ig}A(P') \tag{6}$$

The total phase shift of a particle after it collides with all the other particle on the ring must satisfy the following equation if we consider the periodic boundary condition:

$$e^{ik_nL} = \prod_{m\neq n}\frac{k_n - k_m + ig}{k_n - k_m - ig} \tag{7}$$

Taking the logarithm, we get:

$$k_n = \frac{2\pi I_n}{L} - \sum_m \theta(k_n - k_m) \tag{8}$$

The above equation in fact is Bethe equation [3], where I_i takes value from 1 to N, $\theta(k_n - k_m)$ is the phase shift

$$\theta(k_n - k_m) = 2arctan(\frac{k_n - k_m}{g}) \tag{9}$$

Notice that when $g = 0$, there is no interaction between bosons, $\theta(k_n - k_m) = 2\pi$, the quasi-momentum $k_n = 0$, it represents a ground state for bosons. When $g = \infty$, the particle becomes hard-core boson, $\theta(k_n - k_m) = 0$, the quasi-momentum $k_n = \frac{2\pi I_n}{L}$, this is very similar to the free fermion. Thus, we can get an important conclusion: The interaction will change the statistics between the particles, which is decided by the phase shift in the two particles collision. The bosons can have fermionic characteristic by tuning the interaction, this is the first emergent phenomenon we are talking about in the paper.

This fermionic excitation can be clearly observed in the antiferromagnetic Heisenberg chain, which can also be solved by Bethe ansatz [4]. I will not repeat the calculation process here, but will show some calculation results. Let's first consider the following model:

$$H = \sum_i [S_i^z S_{i+1}^z + \lambda (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)] \tag{10}$$

When $\lambda = 0$, it is Ising model, the ground state is Neel order. When we increase λ, the fluctuation in the second term will partially distroy the order. The spinon will start to propagate in the short range (It is in a confined phase, where the spinons forms bound state). When $\lambda = 1$, it is antiferromagnetic Heisenberg model, the ground state is on a critical point, with an algebraic order. At this point, the spinons can move freely in the chain and totally destroy the long range order. The ground state can be considered as the condensation of spinons. The excitation here is free spinons, which has be oberseved in the experiments as shown in Fig.(2). The nontrivial thing here is that spinons are fermions, while the original system is a bosonic model.

In the above calculation, we used Bethe ansatz, which can be used to exactly solve some one dimensional system, where the interaction can be decomposed into successive two body interaction. The key ideas here can be concluded in the Yang-Baxter equation [5], which is shown in Fig.(3).

Besides the fermionic excitation, there is also local bosonic excitation in these models, no matter the original model is composed by fermions or bosons. This bosonic excitation is coherent in the low energy area and can be understood in the language of bosonization. The basic idea is like this: consider the free Dirac fermion in $1+1$ dimensions, the density operator satsifies the following algebra:

$$\begin{aligned} [j_0(x), j_0(x')] &= [j_1(x), j_1(x')] = 0 \\ [j_0(x), j_1(x')] &= -i\partial_x(\delta(x - x')) \end{aligned} \tag{11}$$

Since the axial current

$$j_\mu^5 = \overline{\psi}\gamma_\mu\gamma^5\psi \tag{12}$$

is conserved, we can define the bosonic field ϕ, which satisfies $j_0 = \partial_x\phi$, $j_1 = -\partial_t\phi = -\pi$ and $\partial^2\phi = 0$. If we consider the interaction $(R^\dagger R)^2$ and

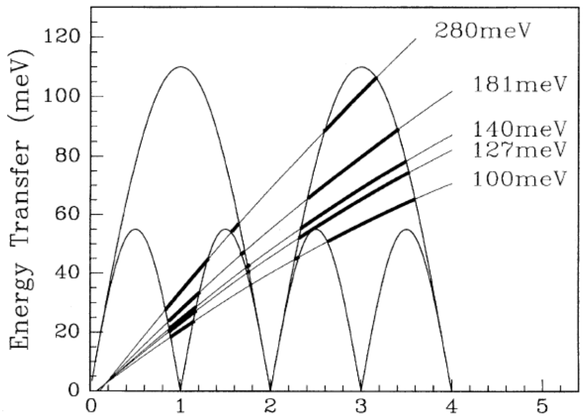

Figure 2: Two-spinon form incoherent excitation spectrum in $KCuF_3$ [12], which is different from magnon excitation

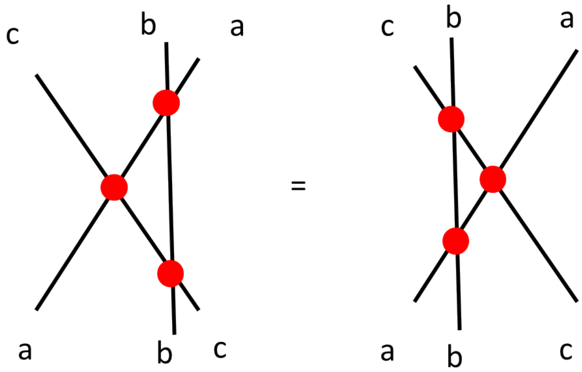

Figure 3: Yang-Baxter equation $Y_{ab}Y_{ac}Y_{bc} = Y_{bc}Y_{ac}Y_{ab}$. The scattering can be decomposed into product of two-body collision.

$R^\dagger L^\dagger RL$, the axial current is still conserved, the Hamiltonian in the density operator representation has this form [4,6]:

$$H = \lambda j_0^2(x) + \frac{1}{\lambda} j_1^2(x) \tag{13}$$

The ground state for this Hamiltonian is

$$\psi = exp(\lambda \int\int j_0(x) log|x - x'| j_0(x') dx dx') \tag{14}$$

with $H|\psi >= 0$. The corresponding Lagrangian for this Hamiltonian in the language of bosonic field is

$$L = \lambda \partial_\mu \phi \partial^\mu \phi \tag{15}$$

which in fact is a free boson system and indicates the fermionic system in $1+1$ dimensions can have coherent bosonic excitation. This is the second emergent phenomenon in this paper.

Another interesting model I want to mention here is the Calogero-Sutherland model[7], which takes this form:

$$H = -\sum_i \frac{\partial^2}{\partial x_i^2} + 2(\frac{\pi}{L})^2 \sum_{i<j} \frac{\lambda(\lambda-1)}{sin^2[\pi(x_i - x_j)/L]} \tag{16}$$

The ground state for this Hamiltonian is

$$\psi_0 = \prod_{i<j} |sin[\pi(x_i - x_j)]/L|^\lambda \tag{17}$$

This is because the Hamiltonian can be written in this form $H = \sum_i Q_i^\dagger Q_i + E_0$, with $Q|\psi >= 0$, where

$$Q = -\frac{\partial}{\partial x_i} + \frac{\pi}{L} \sum_j \lambda cot[\pi(x_i - x_j)/L] \tag{18}$$

The ground state can be written in the complex coordinate

$$\psi_0 = \prod_{i<j} (z_i - z_j)^\lambda \prod_{i=1}^{\lambda} z_i^{-(N-1)\lambda} \tag{19}$$

The excitation around the ground state has this form:

$$\psi = \phi \psi_0 \tag{20}$$

where ϕ is the eigenfunction for the Laplace-Beltrami operator:

$$L = \sum_{i<j} (\frac{z_i + z_j}{z_i - z_j})(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j}) \tag{21}$$

Notice that operator L is not Hermitian. The eigenfunction for this operator is Jack polynomial. This operator squeezes a pair $z^{k_i} z^{k_j}$ into $z^{k_i + l} z^{k_j - l}$, which has great application in fractional hall effect [8]. We've known before that these one dimensional model can be understood in the interacting fermion or boson language. In the following, I will focus on the interacting fermion picture. In this picture, the energy spectrum can be written in this form:

$$E(I) = (\frac{2\pi}{L})^2 \sum_I I^2 n_I^2 + (\lambda - 1) \sum_{I<I'} (I' - I) n_I n_{I'} \tag{22}$$

where I is integer and can take value from 1 to N. The first term in Eq.(22) is free fermion energy and the second term is interacting energy. n_I can only take value 0 or 1. By reorganizing Eq.(22), we can get:

$$E(k) = \sum_i k_i^2 \tag{23}$$

with

$$k_i L = 2\pi [I_i + (\lambda - 1) \sum_j sgn(k_i - k_j)] \tag{24}$$

when $\lambda = 0$, it becomes free bosons, when $\lambda = 1$, it becomes free fermions. When λ takes other values, the system is interacting fermion or bosons. For instance, when $\lambda = 3$, there are two zeros between every two neighboring particles in the ground state configuration. It seems that due to the repulsive interaction between the particles, they will expel from each other. This is the same as the Lieb-Liniger model we mentioned before: the interaction will dramatically change the statistics between particles. Haldane generalized these models and proposed the general Pauli exclusion principle [9]. In this case, the interacting fermion/boson in the Calogero-Sutherland model can be considered as free anyon, which satisfies the general pauli principle.

The general Pauli principle can be understood in this way: for a many particle state, the number of adding a new particle depends on the original many particle state. For instance, the total one particle state for a system has number G. If the systems are free bosons, the number of state of adding a new particle is $D = G$. If the systems are free fermions, the number of available state for adding a new particle depends on how many states has been occupied, thus the number of state for this new particle equals to

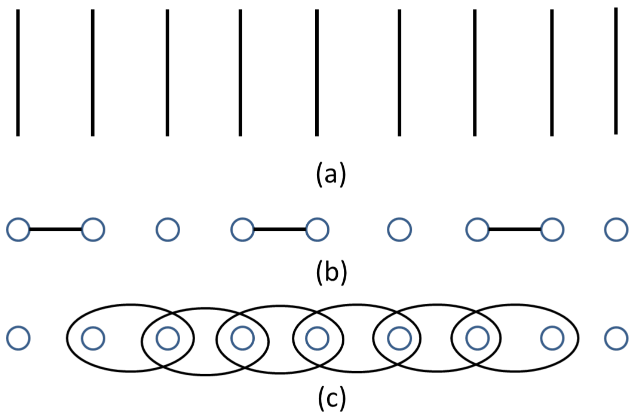

Figure 4: (a) Lowest Landau level in the Landau gauge.(b) Charge density wave phase at 1/3 filling. (c) the one dimensional fractional hall liquid, which is similar as AKLT model.

$D = G - (N-1)$ (The original system has $N-1$ particles). In general, we can define the relation between D and N:

$$\Delta D = -\lambda \Delta N \tag{25}$$

where λ is the statistical parameter. This is definition for the anyon from the aspect of statistical mechanics. The possible state for the N particle free anyon gas is

$$W = \frac{(D+N-1)!}{N!(D-1)!} \tag{26}$$

Using the basic knowledge in the statistical mechanics, we can derive the thermaldynamics for the free anyon gas [10]. The statistical distribution satisfies:

$$n(\epsilon) = \frac{1}{w(\epsilon)+g} \tag{27}$$

where $w(\epsilon)$ is the solution of the following equation:

$$w^{\lambda}(1+w)^{1-g} = e^{\beta(\epsilon-\mu)} \tag{28}$$

For the anyons with general Pauli principle, they always have some fermionic characteristics, and have the pseudo-fermi surface. For the Calogero-Sutherland model, the ground state is the occupied state up to the pseudo-fermi point. There are three possible excitation: particle excitation, hole excitation, particle-hole excitation. For the particle excitation, we just add a new particle to the system with the fermi point shifted a little. For instance, when $\lambda = 3$, it has:

$$|1001001> \rightarrow |1001001001> \tag{29}$$

For the hole excitation, non-trivial things happen, it is similar as the domain wall excitation:

$$|1001001> \rightarrow |10010001> \tag{30}$$

with the effective charge $-1/\lambda$. Notice that the original system is composed of particle with integer charge. The fractional charge excitation is the emergent phenomenon due to the interaction between the particles.

After discussing several one dimensional interacting models, let's move to the two dimensional fractional hall liquid, which has close relationship with these interacting one dimensional models.

The integer Hall liquid is a non-interacting system, in the Landau gauge, the ground state looks like the pattern in Fig.(4)a, which is composed of one dimensional line and fully occupies the two dimensional plane. The electron is bounded in the x direction but can move freely in y direction. When we add a interaction term between particles, since the plane is fully filled, nothing interesting will happen.

The interesting things happened when the plane is partially filled. In the non-interacting limit, the ground state is highly degenerate by picking up randomly M states from N states. When we add interaction, at odd filling factor, the ground state will open a gap and form a bound state without any symmetry breaking. According to Fig.(4)a, the two dimensional model with pseudo-potential interaction can be mapped to a effective one dimensional model[11]:

$$H = g\kappa^3 \sum_p b_p^\dagger b_p \tag{31}$$

where κ is the radio between the magnetic length and the circumference of the cylinder $\kappa = 2\pi l_B/L$ and b_p

$$b_p = \sum_q (qe^{-\kappa^2 q^2} c_{p-q} c_{p+q}) \tag{32}$$

The physical picture for the above model can be understood in the following toy model (at 1/3 filling):

$$H = H_1 + H_2 \tag{33}$$

with

$$H_1 = \sum_i n_i n_{i+1} + \frac{1}{2} n_i n_{i+2} \tag{34}$$

$$H_2 = g \sum_i c_i^\dagger c_{i+1} c_{i+2} c_{i+3}^\dagger \tag{35}$$

H_1 is repulsive potential and H_2 is pair-hopping term. In the thin-torus limit $l_B >> L$, $\kappa = 0$, the Hamiltonian reduces to H_1 with $g = 0$, the ground state is charge density wave with three different ground states on the torus $|100100100... >$, $|010010010... >$, $|001001001... >$. For the two dimensional limit which corresponds to the fractional hall liquid phase, $l_B << L$, $g = 1$. When we increase g from 0 to 1, the charge density wave phase and fractional hall liquid phase are adiabatically connected without closing the gap. The two phases are plotted in Fig.(4)b and Fig.(4)c. These two phases all have 1/3 fractional charge excitation. This method can be generalized to other state in fractional hall liquid.

In conclusion, we have discussed three emergent phenomena in low dimensional strongly correlated system. The first emergent phenomenon is the changing from boson to fermions by tuning the interactions, which can be generalized to the concept of anyons obeying general Pauli exclusion principle. The second emergent phenomenon is the local coherent bosonic excitation in low energy excitation which can be understood in the language of Luttinger liquid. The third emergent phenomenon is the fractional excitation in the system composed of integer charge, which occurs in the fractional hall liquid and some one dimensional correlated system.

[1]K. Ueno, M. Farzaneh, S. Yamaguchi and H. Tsuji, arXiv:1102.4890.

[2]Elliott H. Lieb and Werner Liniger, Physical Review 130: 1605-1616, 1963.

[3]H. A. Bethe, Z. Phys. 71, 205 (1931).

[4]E. Fradkin. Field Theories of Condensed Matter Systems, Addison-Wesley.

[5]V. Chari and A. Pressley, A Guide to Quantum Groups, (1994), Cambridge University Press, Cambridge.

[6]E. Fradkin and E. Moreno, arXiv:9207003.

[7]B. Sutherland, Beautiful models: 70 years of exactly solved quantum many-body problems, (2004) World Scientific Press.

[8]B. Bernevig and F.D.M. Haldane, arXiv:0707.3637.

[9]F.D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991).

[10]M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. 73, 3331 (1994).

[11]D.H. Lee and J.M. Leinaas, arXiv: 0309260

[12]D. Tennant, R. Cowley, S. Nagler, A. Tsvelik, Phys. Rev. B. 52, 13368 (1995)