Particle-Hole Symmetry in the $\nu = \frac{5}{2}$ Fractional Quantum Hall State, or, Is the Glass Half-empty or Half-full?

Physics 569 Term Paper
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Fall 2011

Introduction

The quantum hall effect (QHE) has been one of the richest sources of interesting new phenomena to study in condensed matter physics for the past thirty years. Originally observed by Klaus von Klitzing in 1980 when he was studying anomalies in the Hall resistance of inversion layers, or interfaces between oppositely doped semiconductor samples, in strong magnetic fields (Prange and Girvan, 1990). What he unexpectedly found late one February night in the lab was that there were plateaus in the Hall resistance measured as a function of gate voltage. Not only that, but the behavior was surprisingly universal: regardless of the sample type, geometry, or material, the plateaus occurred at very particular values. These special values of the Hall resistance are

$$R_H = \frac{h}{ne^2}$$

where h is Planck's constant, e is the magnitude of the electron charge, and n is a small integer.

Given this quantization of the Hall resistance by integer values, the phenomenon came to be called the Integral Quantum Hall Effect (IQHE). As we will see, it is the two dimensional constraint on the electron system in between the different semiconductors that leads to the novel physical effects – completely unexpected prior to their experimental discovery! It is a small wonder that von Klitzing won a Nobel prize in 1995 for his pioneering work. Briefly, the IQHE can be explained in terms of non-interacting electrons confined to two dimensions where the strong transverse magnetic field causes a quantization of the kinetic energy states into degenerate harmonic-oscillator-like states called Landau levels (LL), after his theoretical prediction of them many years before. It is this energy eigenstate structure, combined with the localization effects of impurities and disorder in the samples, that causes the plateaus in the Hall resistance (Prange and Girvan, 1990). We shall return to discuss LL in more detail, but we will not go into more detail about the explanation of the IQHE. For now, it is useful to note that the integer n is actually given by the *filling factor*, or number of LL fully occupied by electrons.

Much to everyone's surprise, just two years later the range of Quantum Hall phenomena exploded with the discovery of the Fractional Quantum Hall Effect (FQHE). Tsui, Stormer, and Gossard discovered in 1982 a plateau at a Hall resistance of $R_H = 3h/e^2$, i.e. they found that the IQHE generalized to

$$R_H = \frac{h}{\nu e^2}$$

for $\nu = 1/3$, though many other rational number plateaus were soon to follow. Thus, we observe also quantization of the Hall resistance at *fractional* filling factors – generally fractions less than one, meaning the lowest Landau Level (LLL) is only partially filled. The IQHE can be explained in terms of non-interacting electrons, but a theoretical description of the FQHE cannot neglect such complications. It is important to note that though the electrons are confined to a thin region between two semiconductors, and can be considered a two-dimensional quantum system, their interactions still use all three dimensions and are given by a standard Coulomb repulsion. In principle, we can say all there is to say about the microscopic situation by writing a Hamiltonian including all the kinetic energies and two-body interactions, but as is often the case, "More is Different." In particular, we cannot perturb from a non-interacting "normal state," and recover the qualitative effects that are observed – we are in a "strongly non-perturbative" regime (Jain, 2007). We shall later discuss the explanation of the FQHE in terms of "composite fermions," or electrons attached to quantum vortices, which provide a nice analogy between the IQHE and FQHE.

In this paper, we will be particularly interested in the $\nu = 5/2$ state (Willett et.al., 1987; see Fig. 1), which has many interesting properties not found in many (if any at all) other QHE states. In particular, it is believed to be accurately described by a p-wave paired state of composite fermions, and given by a Pfaffian wave function as first guessed by Moore and Read (1991):

$$\Psi_{1/2}^{Pf} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}\left(z_i - z_j\right)^2 e^{-\frac{1}{4}\sum_k |z_k|^2}$$

where Pf denotes the Pfaffian, or square root of the determinant, of the matrix enclosed, and the z_j are the complex coordinates $z = x - iy$ of the electrons (the reason for the sign will become apparent later). We used the subscript 1/2 on the wave function, instead of 5/2, because this actually describes a half-filling of the second LL. The LLL is doubly filled by opposite spin electrons, hence the total filling factor is 5/2, but in the ideal case the LLL can be considered inert (Jain, 2007). The Pfaffian of an antisymmetric matrix M is automatically antisymmetric itself, given by

$$\mathrm{Pf}M_{ij} = A(M_{12}M_{34}\ldots M_{n-1,n})$$

where A denotes the antisymmetrization operation.

This Pfaffian state has interest at a more fundamental level than other FQHE states because it is one of the only known natural system in which it is possible to have non-abelian anyons – quasiparticle excitations whose statistics differ from the standard dualism of bosons/fermions. In two dimensions, particle exchanges turn out to be a representation of the braid group instead of the permutation group, due to the topological inequivalence of the different possible trajectories – how we exchange particles turns out to be as important as the fact that they've been exchanged, which is all the permutation group tells us. Abelian anyons obey statistics based on a one dimensional unitary representation of the braid group, i.e. complex numbers of magnitude one. Abelian anyons have statistics given by $e^{i\phi}$, and hence are more general than fermions and bosons, which correspond to the particular cases of $\phi = \pi$ and 0 respectively. Higher dimensional representations involve matrices which do not necessarily commute, and hence are called non-abelian statistics. These possibilities have not yet been experimentally observed, but it is exciting to think that nature could contain such strange objects.

On the technological front, non-abelian anyons are being investigated as a possible means of constructing a quantum computer. We mentioned that anyon exchange operations form a representation of the braid group, and it is theoretically possible to simulate any unitary quantum computation gate to arbitrary accuracy by combining different braid operations. This would be implemented by physically exchanging the particles, but due to the topological nature of the operation being performed (i.e., variations in the trajectories do not matter so long as they are topologically equivalent to the braid group elements desired) such computations would be far more fault-tolerant than standard quantum computation methods. Such a system of braided non-abelian anyons is called a topological quantum computer (TQC).

Hence, the $\nu = 5/2$ state is a possible physical implementation of a TQC, so it is well worth investigating its properties. This paper will take a look at the issue of particle-hole symmetry. A particle-hole transformation is antiunitary, and corresponds to replacing all creation operators with annihilation operators (and vice versa) in the Hamiltonian (Lee et. al., 2007). The standard two-body Coulomb interaction, which is the only relevant microscopic effect present in QHE systems and hence must underlie the myriad emergent phenomena observed, is particle-hole symmetric. On the other hand, the Pfaffian state is known to be the exact ground state of a repulsive three-body Hamiltonian that breaks particle-hole symmetry (Greiter et. al., 1991). Therefore, the $\nu = 5/2$ state must spontaneously break particle-hole symmetry, if the Pfaffian wave function is indeed the description of it. This is the subject we shall endeavor to investigate, but first we shall review the subjects of Landau levels and fractional quantum hall states.

Landau Levels

The concept of Landau levels is at the heart of the QHE, so it is important to understand them. The ideal system from which they arise is that of nonrelativistic charged particles confined to the x-y plane, interacting with a constant magnetic field pointing the z-direction. Following the treatment by Jain, we introduce the vector potential $\mathbf{A}$, such that $\mathbf{B} = \nabla \times \mathbf{A} = B\hat{\mathbf{z}}$, and therefore write the free electron Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{A}}{c}\right)^2 .$$

We note that with this Hamiltonian the time independent Schrödinger equation $H\Psi = E\Psi$ is invariant under gauge transformations of the form

$$\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + \nabla\chi(\mathbf{r})$$
$$\Psi(\mathbf{r}) \to e^{-ie\chi(\mathbf{r})/\hbar c}\Psi(\mathbf{r}) \quad ,$$

hence we are free to choose a gauge which makes our calculations convenient, since no observable quantities can depend on the gauge.

We shall choose the symmetric gauge $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$, which we can verify returns the correct magnetic field upon taking the curl. This gauge is chosen, instead of the simpler Landau gauge $\mathbf{A} = -By\hat{\mathbf{x}}$, because it is useful for expressing our positions with complex coordinates. We define $z = x - iy = re^{-i\theta}$ and its complex conjugate $\bar{z} = x + iy = re^{i\theta}$, and then re-express the partial derivatives with respect to x and y in terms of z and $\bar{z}$ in order to rewrite the Hamiltonian in position space as

$$H = \frac{1}{2}\left[-4\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4}z\bar{z} - z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right]$$

where all quantities are in units of the natural length and energy scales: the magnetic length $\ell = \sqrt{\hbar c/eB}$, and the cyclotron frequency $\hbar\omega_c = \hbar eB/mc$. Next we define the ladder operators

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right)$$
$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right)$$
$$b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right)$$
$$b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right)$$

in terms of which the Hamiltonian becomes

$$H = a^\dagger a + \frac{1}{2} \ .$$

Thus we see the problem is equivalent to a harmonic oscillator, since $[a,a^\dagger] = [b,b^\dagger] = 1$, and the energy eigenvalues are given by $E_n = \hbar\omega_c(n + 1/2)$ where n is the number of quanta created by $a^\dagger$ acting on the ground state. These quantized energy levels are the Landau levels, but they are degenerate since we can also write the angular momentum operator

$$L = -i\hbar\frac{\partial}{\partial\theta} = -\hbar(b^\dagger b - a^\dagger a)$$

which commutes with H and has eigenvalue spectrum $m = -n, -n+1, \ldots$, so we can write our complete basis of non-interacting states as

$$|n,m\rangle = \frac{(b^\dagger)^{m+n}}{\sqrt{(m+n)!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle .$$

Where we had to raise $b^\dagger$ to the $(m+n)^{\text{th}}$ power since $a^\dagger$ lowers the L eigenvalue. Finally, we write the LLL eigenstates

$$\psi_{0,m}(\mathbf{r}) = \langle \mathbf{r}|0,m\rangle = \frac{z^m e^{-\frac{1}{4}z\bar{z}}}{\sqrt{2\pi 2^m m!}}$$

which can be derived by solving for the ground state that is annihilated by both a and b, then acting on it m times with $b^\dagger$. Hence a general LLL state is some superposition of these eigenstates, and will be an arbitrary analytic function of z, justifying our sign convention earlier.

In an infinite planar geometry, the degeneracy of each Landau level is infinite, but it turns out that the degeneracy *per area* is a constant (independent of gauge choice, which in general affects the eigenstates we will get). In the LLL of the symmetric gauge we can see that $|\psi_{0,m}|^2$ is peaked at $r = \sqrt{2m}\ell$, hence a disk of finite radius R will contain (to first order) $m = R^2/2\ell^2$ degenerate states per LL, or a degeneracy per unit area of $1/2\pi\ell^2$. Therefore, since the filling factor is the number of electrons per accessible states in a LL, we can write it in terms of the electron density as $\nu = 2\pi\ell^2\rho$. Then from the definition of the magnetic length we can see that increasing the B-field decreases the filling factor.

Now the notion of a filled LL is well-defined – it is when all the degenerate states are occupied by electrons. We note that due to the strong magnetic field present, there is generally spin polarization, so we fill all m states with identical spin electrons, and arrive at a filling factor of $\nu = 1,2,\ldots$ etc., and the total wave function Φ_ν is given by antisymmetrizing the m single particle wave functions with a Slater determinant. In particular, for a single filled Landau level we have

$$\Phi_1 = \prod_{j<k} (z_j - z_k) e^{-\frac{1}{4}\sum_k |z_k|^2} .$$

Fractional Quantum Hall Effect and Composite Fermions

We have derived the Landau levels of non-interacting electrons in a transverse magnetic field, and noted earlier that IQHE states correspond to integer filling factors ν. We shall now take this "explanation" of the IQHE as a given, without going into more details about the relation between these states and the observed Hall resistance plateaus, and explain the analogy made between the IQHE and FQHE.

The essential concept (following the treatment of Jain once again) used is that of a composite fermion, which is a bound state of an electron and an even number of quantum vortices. In this context, "vortex" means a location around which a closed loop changes the phase of the wave function by 2π, so e.g. for $\psi \propto (z_1 - z_2)^{2p}$, each particle "sees" $2p$ vortices on the other. FQHE vortices occur in the wave function, in contrast to the occurrence of vortices in the order parameter of a superconductor. Hence, instead of representing a magnetic flux tube around which supercurrents circulate, there is no real flux associated with a FQHE vortex, and the physical magnetic field is given by the external applied field. However, the composite fermions interact with an effective magnetic field which is increased/decreased (depending on orientation) from the external field by the vortices.

We now make the analogy between IQHE and FQHE. Starting from a system of non-interacting electrons in a

$$\nu^* = 2\pi\ell^2\rho = \frac{\rho\phi_0}{|B^*|} = n$$

IQHE state, where $\phi_0 = hc/e$ is the magnetic flux quantum and B^* is the external magnetic field, we then attach $2p$ flux quanta (i.e. vortices) to each electron. Next a mean-field approximation is made, whereby the flux is spread out to become part of the uniform magnetic field, so the composite fermions experience a field

$$B = B^* + 2p\rho\phi_0 = \pm\frac{\rho\phi_0}{n} + 2p\rho\phi_0$$

so the electron system will be at filling factor

$$\nu = \frac{\rho\phi_0}{B} = \frac{n}{2pn \pm 1}$$

and we have fractional quantum hall states! This is a very heuristic description of a topological ordering of states that is given in full detail by a Chern-Simons field theory (Lopez and Fradkin, 1998). It is important to note that in general the FQHE is understood as a topological order, and not an order of spontaneous symmetry breaking.

At this point, we can understand the $f = 1/m$ states first given by Laughlin in 1983. The Laughlin wave functions are

$$\begin{aligned}\Psi_{1/m} &= \prod_{j<k}(z_j - z_k)^m e^{-\frac{1}{4}\sum_k |z_k|^2} \\ &= \prod_{j<k}(z_j - z_k)^{2p}\Phi_1\end{aligned}$$

where $m = 2p + 1$. That is, we have attached $2p$ vortices to each electron in a single filled Landau level. Notice that since the above wave function is analytic in z, it is automatically a LLL state. In general, when we start with $n > 1$, the non-interacting wave function Φ will not be analytic, so after "attaching" the vortices by multiplying by the $(z_j - z_k)^{2p}$ factor (known as a "Jastrow" factor), we will have to project the wave function down into the LLL and remove the non-analytic parts.

The reader may have noticed already that there is something fishy about the kind of fraction given above: it only admits an *odd* denominator, but we have said we're interested in the 5/2 state which has a decidedly *even* denominator. However, in the limit that $n \to \infty$, we see that $f = n/(2pn+1) \to 1/2$ (Halperin, Lee, and Read, 1993). This is deceptively simple: first of all, there is no observed FQHE state at $f = 1/2$; second of all, it is not at all clear that we can increase n without limit. For example, in the ninth Landau level, there isn't a stable FQHE state (i.e. for $f = 16 + n/(2pn + 1)$) (Jain, 2007). We also see that $n \to \infty$ implies the effective field $B^* \to 0$, so instead of Landau-like levels filled with composite fermions, we expect a composite-fermion-sea, exactly analogous to the standard fermion sea.

Thus, to have even-denominator fractions, we must perturb this state with weak interactions between the composite fermions, which we have so far treated as non-interacting (do not confuse the interactions between electrons, which lead to the composite fermion picture, with interactions between the composite fermions, which by virtue of being mapped onto Landau-like levels are assumed to not interact with each other). This is exactly what happens for the 5/2 state, where the composite fermions undergo p-wave pairing in the half-filled second LL. This can be seen by analogy with the BCS wave function for fully polarized electrons (suppressing the symmetric spin part of the wave function)

$$\Psi_{BCS} = A[\phi_0(r_1,r_2)\phi_0(r_3,r_4)\ldots\phi_0(r_{N-1},r_N)]$$

which by comparing to the formula above we can see is a Pfaffian (deGennes, 1989).

An immediate question this presents is how to get pairing, which requires an attractive interaction, out of underlying Coulombic repulsion. However since the pairing is between the *composite fermions*, instead of the electrons, it is theoretically possible to get a weak attraction if the vortices (over-)screen out the Coulomb repulsion (Jain, 2007). Supporting this view is the fact that experiments have detected composite fermion Fermi sea properties at 5/2 after raising the temperature so as to eliminate the FQHE (Willett et. al., 2002). Therefore, we can think of there being a composite fermion sea which is unstable to pair formation, much like in BCS theory. However, this is not an example of superconductivity of composite fermions: the 5/2 state has *no* off-diagonal long-range order (Jain, 2007).

Even without a microscopic understanding, numerical calculations of the overlap between the Pfaffian state and the exact Coulomb ground state provide good evidence that the description is correct. For example, depending on how the pseudopotential in the Coulomb Hamiltonian is tweaked (i.e. increased or decreased from its real value by some overall numerical factor) the ground state overlaps with the Pfaffian state by as much as ~0.99 (Morf, 1998; see Fig. 2). Despite the numerical evidence, there is as of yet no experimental evidence in favor or against identifying the Pfaffian with the 5/2 state. The Pfaffian is therefore at the very least a very good model for the 5/2 state, so we should seek to understand it as best we can and perhaps find out how to modify it to make it more accurate.

Particle-Hole Symmetry Breaking in the ν=5/2 state

We have thus far discussed what the QHE is in general, where it arises from, and some of the open questions relating to the 5/2 state. We have said already that the Pfaffian state is particle-hole asymmetric. This means that there is a particle-hole conjugate state, the so-called anti-Pfaffian (Levin et. al., 2007; Lee et. al., 2007). The idealized 5/2 state, without any Landau level mixing (i.e. no particle-hole excitations), is exactly symmetric, and in this limit the Pfaffian and anti-Pfaffian become degenerate. This makes sense: ideally, at half-filling, it should not matter whether we have half-filled the second LL, or half-emptied it – the many-body wave function would be the same. However, the sorts of three-body interactions of which the Pfaffian is the ground state are generated in second-order perturbation theory from LL mixing which breaks this symmetry (Levin, 2007). In this case, though, it is not clear whether the Pfaffian or the anti-Pfaffian state should be preferred. Is it possible to detect which comes closer to describing the experimental aspects of the 5/2 state?

It turns out that the two states have qualitatively different edge theories, and thus correspond to different universality classes (Levin, 2007). The Pfaffian has two edge modes: a clockwise (for *B* in the positive z-direction) propagating Majorana fermion, and a clockwise propagating chiral boson. The anti-Pfaffian has three edge modes: counter clockwise propagating Majorana fermions and chiral bosons, along with a clockwise chiral boson. The thermal Hall conductance K_H (heat conductance under an applied temperature gradient) is given, in appropriate units of $\pi^2 k_B^2 T/3h$, by simply counting these modes: for the Pfaffian we get 1/2 + 1 = 3/2 (a half for the fermion, one for the boson, both positive for clockwise propagation), while for the anti-Pfaffian we get –1/2 – 1 + 1 = –1/2. Since thermal conductance is a macroscopic thermodynamic phenomena, this difference means the two states belong to different universality classes (Levin, 2007). More importantly, such a difference should be experimentally detectable, and could decide which state is actually favored by nature (Lee, 2007). That is, we should be able to decide whether the 5/2 glass is half-empty or half-full!

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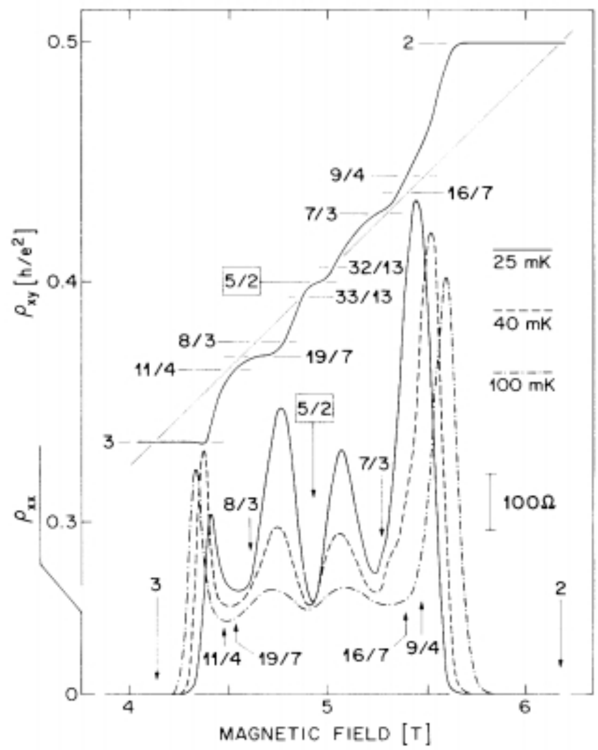

Fig. 1 – Two signatures of QHE state at 5/2: plateau in Hall resistance, and vanishing transverse resistance. Willett et. al., 1987

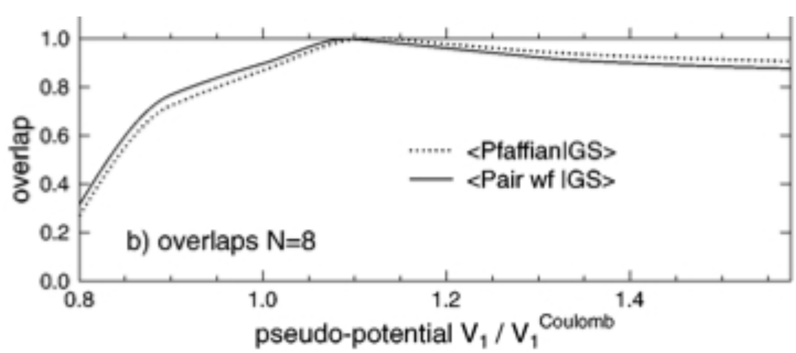

Fig. 2 – Overlap of Coulomb Hamiltonian ground state with Moore-Read Pfaffian state as function of pseudopotential