Ginzburg-Landau theory of the fractional quantum Hall effect

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Abstract

We present a description of the fractional quantum Hall effect (FQHE) in terms of a field theory analogous to the well-known Ginzburg-Landau theory of superconductivity. The key contribution of this theory is that it describes the phenomenology of the FQHE in terms of the behaviour of a charged superfluid. Hence, it provides a unifying understanding of the fractional quantum Hall state in the context of a well-known condensed matter system.

1 Introduction

Solid state systems can exhibit various types of interesting collective behaviour. Superconductivity, magnetism and the Hall effects are a few examples of this. In dealing with such a variety of phenomena, we are faced with the task of understanding the physics from as many view points as possible. The more we can shed light on the different sides of a system, the more we can construct a coherent understanding. In many cases, we can construct two types of theories, namely those that go bottom-up and those that take the top-down approach. Both, I believe, are equally important.

A great example of a system for which we can construct a microscopic as well as an independent macroscopic description is the superconducting state. The BCS variational wave function and the Ginzburg-Landau free energy are both useful, insightful and complementary ways of explaining conventional superconductivity.

The fractional quantum Hall effect (FQHE) is another example for which this agenda was reailzed. Here too an effort was made to come up with both a microscopic as well is a macroscopic theory. The first breakthrough came with a variational approach that lead to a microscopic understanding, namely Laughlin's variational state. A few years after, condensed matter physicists tried to come up with a Ginzburg-Landau-type theory that could provide another complementary point of view. A point of view that could reveal other fundamental aspects of this ver exotic quantum state.

In this essay we try to present one of such attempts, namely the so-called Chern-Simons-Ginzburg-Landau (CSGL) theory. Its objective is to view the FQHE as a charged superlfluid. We will thus see how phenomena from superconductivity and superfluids coalesce to describe the physics of the fractional quantum Hall states.

2 The fractional quantum Hall effect

2.1 The beginning

The FQHE is sustained by the nontrivial combination of single-particle quantum effects, interparticle interactions and dimensionality. However, before we delve into the description of the full FQHE, let us briefly mention what the integer effect entails. This effect is, historically, the precursor of the FQHE.

The integer quantum Hall (IQHE) effect emerges when we subject an electron gas in two dimensions to a sufficiently strong external magnetic field B and low enough temperatures. In this limit, the experimenter can measure the conductivity of the system and discover two important facts. First, the diagonal resistivity ρ_{xx} goes essentially to zero, which signals dissipationaless transport in the system. Second, the Hall resistivity, namely ρ_{xy}, shows extremely well defined plateaus for signifcant ranges of B-field values. This is leads to a quantization of the Halll conductance given by $\sigma_{xy} = ne^2/h$, where n is an integer, e is the electron charge and h is Planck's constant.

This remarkable effect can be explained in terms of quantum-mechanical single-particle physics [9]. If the magnetic field is strong enough, the energy spectrum is largely dominated by the so-called Landau levels. To each Landau level, there corresponds of the order of $n_B = eB/(hc)$ possible states per unit area (e is the electron charge and h is Plank's constant). Hence, for sufficiently large fields, each Landau level is massively degenerate. As the magnetic field is varied, we change the degeneracy of each level, which is equiva-

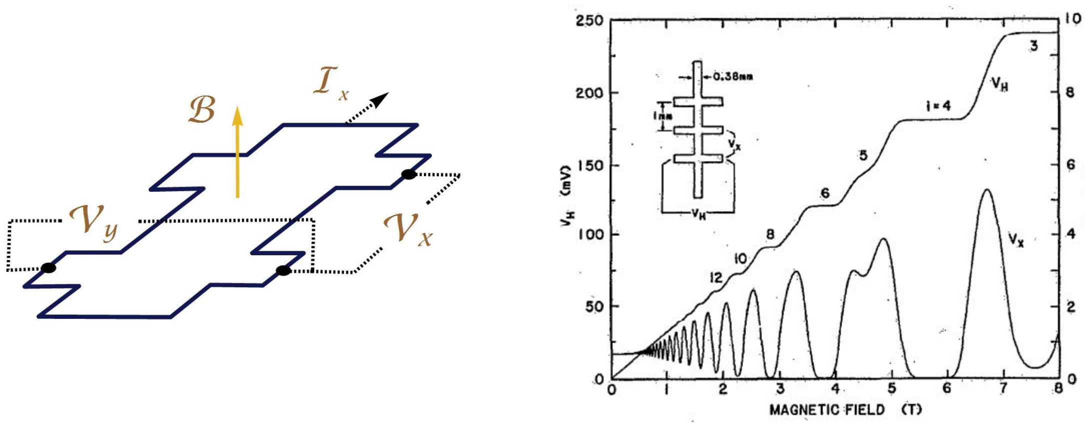

Figure 1: Experimental evidence of the IQHE. **Left:** Schematic representation of the sample. V_x is the potential drop that probes the diagonal resistivity ρ_{xx} and V_y is that which probes the Hall resistivity ρ_{xy}. **Right:** The diagonal and Hall voltages, whcih are proportional to the respective resistivities. The diagonal voltage drops to zero at the range of B-field for which the Hall voltage exhibits plateaus. Taken from [1].

lent to changing the Fermi level of the 2D electrons. Whenever the B-field is such that the electrons fill an integer number of Landau levels, scattering is strongly suppresssed, because there are no states into which the electrons can scatter. Furthermore, when this happens, the number of electrons will be exactly an integral multiple of n_B which implies the quantization of the Hall conductance. Hence, roughly speaking, the structure of the Landau levels serve to explain the appearance of the integer quantum Hall effect.

To explain the stable plateaus, however, it is necessary to include disorder in the system. The presence of disorder, counter to the intuition that it can, for example, destroy dissipationless transport, it is actually responsible for the appearance of the plateaus in the Hall conductivity. Since disorder broadens the Landau level energies, each time we move the Fermi energy through the states in the broadened region, we obtain localized states. Hence, as we vary the magnetic field, and by consequence the Fermi level, the conductivity cannot increase. Only very close to the next exact Landau energy do we get delocalized states again that contribute to the conductivity of the system. Thus, the result is that, as we change the magnetic field, there are necessarily finite spans of B-field values for which the conductivity stays constant.

2.2 From whence come the fractions

The first observation of the FQHE was performed on gallium-arsenide and aluminum-gallium-arsenide heterostructures in 1982 [12]. At that time, the integer quantum Hall states were already generally understood, and it was observed in the same types of systems. The important breakthrough in the 1982 experiment was the low level of disorder that was achieved at that time. This fact, together with low temperatures ($\approx 1K$) and extreme magnetic fields ($\approx 20T$) favoured the observation of the fractional quantum Hall effect, characterized by fractional conductance.

In their breakthrough experiment, Tsui *et al.* they observed, just as in the IQHE, how at a certain fillings of the Landau levels, the diagonal resistivity was suppressed, which

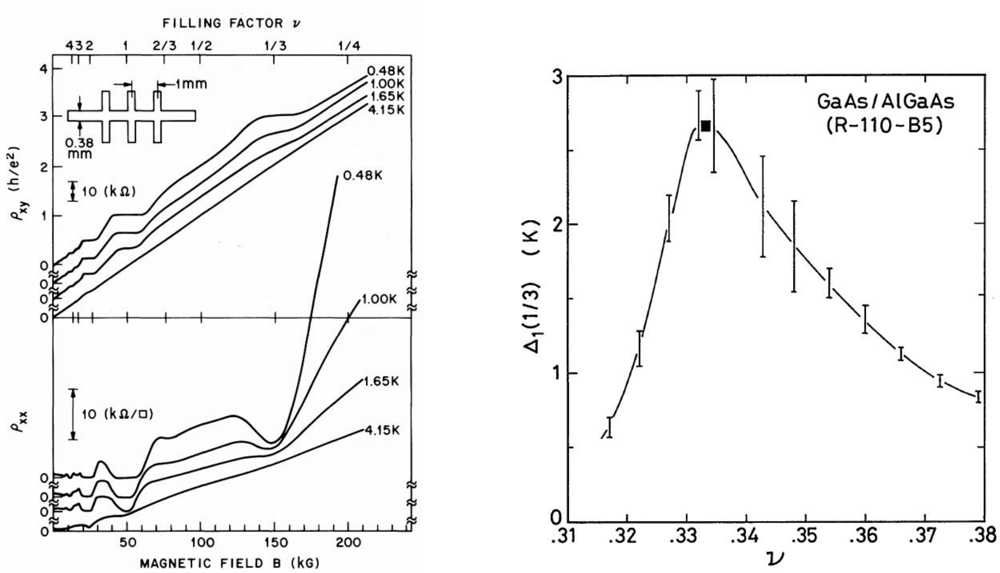

Figure 2: **Left:** The first measurement made by Tsui and collaborators in which they detect a plateau in the Hall resistivity ρ_{xy} and a dip in the diagonal resistivity ρ_{xx} at the unexpected filling $\nu = 1/3$, as a function of the magnetic field. The inset shows the schematic of the sample that was used with the approximate dimensions. Taken from [12]. **Right:** Experimental evidence of the finite amount of energy required to produce thermally activated diagonal resistivity as a function of the filling fraciton. Note that the highest energy occurs precisely near $\nu = 1/3$. Taken from [10].

signaled the appearance of disspationless transport in the sample. Furthermore, they noticed that the Hall resistivity exhibited a plateau. These two features might suggest the emergence of the integer quantum Hall effect.

However, it turned out that this behaviour appeared, not at an integer fillling, but rather at a fractional value. In [12], the first experiment in which this was reported, they concluded that this fraction was $\nu = 1/3$, with approximately 1% accuracy. We show their measurement in the left-hand side of Fig.(2). Subsequent experiments uncovered more nontrivial behaviour. Several additional fractions, such as $2/3, 4/3, 5/3, 2/5, 3/5, 4/5, 2/7$ were observed [4, 11]. The accuracy of the quantization of the 1/3 and 2/3 cases reached 3 parts in 10^5.

Is this then another form of the IQHE? Although the conductivity seems to behave in the same manner, other properties proved to be unique to these states. First, these states showed activated behaviour of the diagonal resistivity [10], depending on the temperature range used, which suggested the presence of an energy gap in the excitation spectrum (we show an example in the right-hand side Fig.(2)). This gap cannot arise from the structure of the Landau levels since the filling is fractional, which means that interaction effects must rearrange the massive degeneracy of the Landau levels [9]. Furthermore, in [2], it was shown how disorder could easily close the excitation gap, which meant that the FQH states are highly sensitive to the disorder in the system.

But perhaps more exotic than these observations was the emergence of quasiparticles with fractional charge and that obey fractional statistics. Having fractional charge means having some fraction of the charge of the electron, which in nonrelativistic systems is not

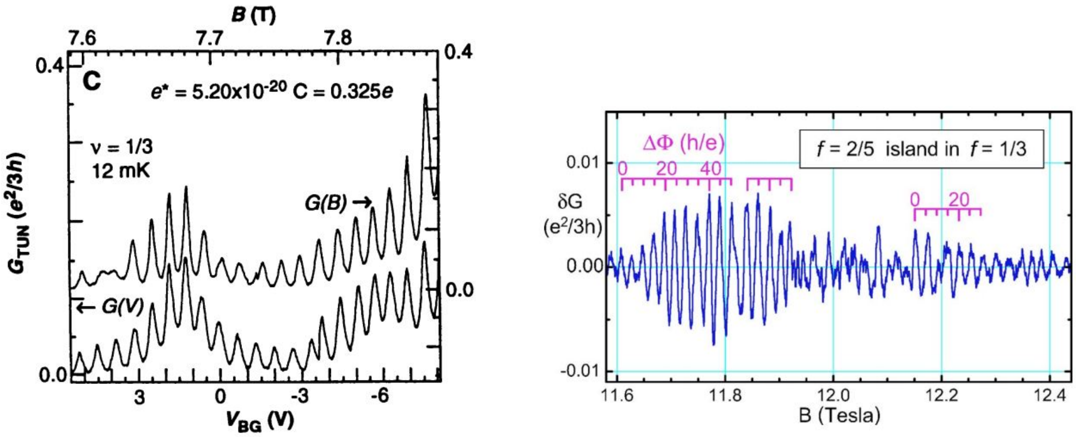

Figure 3: **Left:** Experimental plot in which oscillations in the conductance from resonant tunneling measurements provide a way to determine the quasiparticle charge. Note the number for the measured charge e^* in the figure, close to the value $e/3$. Taken from [6]. **Right:** Experiments in which interferometry was performed on quasiparticles of $\nu = 1/3$ make a loop around an island of FQH fluid of $\nu = 2/5$. This leads to a way of measuring the relative phases gained in the closed path, and thus a way of detecting fractional statistics. Taken from [3].

very common. Fractional statistics refers to how, when two particles are exchanged, an overall phase is gained which is different from the usual bosonic phase (equal to 2π) or the fermionic phase (equal to π). Hence, this effectively changes the statistics of the identical particles, in general by a fractional of 2π [13].

In Fig.(3), we show examples of experimental evidence for fractional charge (left hand plot) and fractional statistics (right hand plot). Admittedly, these more exotic properties were searched for and measured experimentally, only motivated by the theory that was discovered a few years after Tsui's measurements. However, we wanted to mention them in this section to complete the list of basic experimental facts that needs to be explained by any theory of the FQHE.

The purpose of showing the plots is not to give a detailed account of how things were measured, but to provide the reader the feeling that that these exotic objects are, in fact real and detectable. The theory that will be outlined in this essay will account for this and the other important features of the fractional quantum Hall states.

2.3 Bottom-up approach

Before describing the Landau-type theory of the FQHE, let us first review the first theory that explained the FQHE, so that it serves as a comparison point when we discuss the Landau-type description.

The first successful quantitative description of the FQHE was provided by taking a variational approach. In 1983, a year after the first experiments, Laughlin proposed his celebrated variational wave function for the $\nu = 1/3$ state [7]. Laughlin's wave function is given by

$$\psi_\nu(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i \leq j \leq k \leq N} (z_j - z_k)^{\frac{1}{\nu}} \exp\left(- \sum_{j=1}^{N} \frac{|z_j|^2}{4\ell^2} \right) \tag{1}$$

where ℓ is the magnetic length and $\frac{1}{\nu} = 2k + 1$ is an odd integer. This wave function describes the many-body state of fermions with positions labeled conveniently with a complex form of their position $z_j = x_j + iy_j$. Since $\frac{1}{\nu}$ is an odd integer, this function is correctly antisymmetric under the exchange of any two positions.

All the important features of the FQHE we described in the previous section can be obtained from Laughlin's wave function. For example, it is an incompressible state which means that the system avoids density changes when a small amount (an arbitrarily small amount) of pressure is exerted on the system. A finite amount of pressure can eventually promote the creation of quasiparticles in the system. This then explains the existence of a gap in the system at fractional filling and the possibility of dissipationless transport. Furthermore, one can construct explicity the wavefunctions of the corresponding quasiparticle excitations, and prove that they carry fractional charge $\frac{e}{2k+1}$ and fractional statistics.

We would like to specially note that there is an important insight into the microscopic physics that is at work in the FQHE. Halperin noticed from this wave function that there are no "wasted zeros" [8]. Becaue the sample is being pierced by $\frac{N}{\nu} = N(2k + 1)$ flux quanta, the wave function must have the same number of zeros so that the correct Aharonov-Bohm phase is recovered when traversing the sample in a loop. Automatically, due to the Pauli principle, N of these zeros will be attached to the positions of the particles.

However, from Laughlin's wavefunction we see that *all* zeros are in fact attached to the particles, so that each electron spends most of their time away from other electrons. This insight thus uncovers the manner in which electrons minimize the interparticle interaction energy. As we shall see later, this behaviour had bearing on the understanding of off-diagonal long range order in the fractional quantum Hall states that we will discuss later on.

3 Ginzburg-Landau-type theory

Having a description of the FQHE through Laughlin's variational wave function, and having obtained from it a microscopic understanding, it is then reasonable to ask why it would be necessary to seek a low-energy effective theory. The reason, as we mentioned in the introduction, is analogous to what happens in trying to understand conventional superconductivity. Eventhough, in principle, the BCS theory can provide us with the most relevant phenomenology of superconductors, in some cases it is not easy to discern elements of the theory which are part of the fundamental and defining properties of the superconducting state. The macroscopic description, namely the so-called Ginzburg-Landau theory, is able to provide a global picture of the essential physics involved that would otherwise be obscured in microscopic theories.

We will here now describe the path taken by Zhang, Hansson and Kivelson [15] to obtain an effective field theory description of the fractional quantum Hall effect.

3.1 Fermions get disguised as bosons

The key first step is to map *exactly* the system of interacting fermions that make up the FQHE into a system of bosons. We start with the microscopic Hamiltonian, given by

$$H = \sum_i \frac{1}{2m}\left[\mathbf{p}_i - \frac{e}{c}\mathbf{A}(x_i)\right]^2 + \sum_i eA_0(x_i) + \sum_{i<j}\frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{2}$$

The wave functions of this fermionic system are, of course, antisymmetric. To obtain a bosonic system, let us perform a unitary transformation given by

$$U = \exp\left(-i\sum_{i<j}\frac{\theta}{\pi}\alpha(x_i, x_j)\right) \tag{3}$$

where θ is a free parameter which we will fix conveniently so that symmetrized wave functions are obtained, and $\alpha(x_i, x_j)$ is the angle made by the line joining x_i and x_j with respect to one of the axes of the reference frame. Under this transformation, we will get a wave function of the form

$$\psi_B(\{x_k\}) = U\psi_F(\{x_k\}) \tag{4}$$

where the indices stand for bosonic and fermionic, respectively.

Suppose now that we exchange two fermions at positions x_k and x_l. Then ψ_F would acquire a π phase corresponding to antisymmetry in the wavefunciton. In addition to this, such an exchange affects the unitary transformation U as well, since, by the definition, we have that $\alpha(x_k, x_l) = \alpha(x_l, x_k) + \pi$. Thus, ψ_B gets an overall phase of $\pi - \theta$. If we choose $\theta = (2k+1)\pi$, with k an integral, the overall phase change under exchange of particles will be equivalent to 2π. We thus obtain a bosonic wave function by adjusting our free parameter in this way.

This mapping does not, however, come without a cost. The transformation U is a gauge transformation, one which is singular, in the sense that the angle function is not well defined for points which lie on top of each other. Being a gauge transformation, the Hamiltonian must be transformed accordingly. The effect of computing UHU^{-1} is to generate a shift in the kinetic energy term. Thus, in addition to having the vector field due to the external magnetic field, we get an additional field which in the literature is called $\mathbf{a}$.

The transformed Hamiltonian that applies to the effective bosons is then given by

$$H = \sum_i \frac{1}{2m}\left[\mathbf{p}_i - \frac{e}{c}\mathbf{A}(x_i) - \frac{e}{c}\mathbf{a}(x_i)\right]^2 + \sum_i eA_0(x_i) + \sum_{i<j}\frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{5}$$

where $\mathbf{a}(x_i) = \frac{c\hbar\theta}{\pi e}\sum_{j\neq i}\nabla\alpha(x_i, x_j) = \phi_0\frac{\theta}{2\pi^2}\sum_{j\neq i}\nabla\alpha(x_i, x_j)$, with $\phi_0 = \frac{ch}{e}$ the flux quantum. Hence, we conclude that, in order to describe the manybody system as a set of bosons, instead of fermions, we have to introduce an extra magnetic field into the system. Note that this field is explicitly dependent on the coordinates of the coordinates of the particles so that, in a way, it is as if it was attached to them. As we shall see, this field is fundamentally responsible for the physics of the FQHE in this bosonic picture.

Now, having our new Hamiltonian, we would like to obtain a Lagrangian that will yield the same dynamics for the fields of the system. Since the idea is not to get into technical details, but to present the physics the emerges, we will keep this section concise. It is possible to show that the corresponding action and Lagrangian can be written as

$$S = S_\phi + S_a = \int d^3x \mathcal{L}_\phi + \int d^3x \mathcal{L}_a \tag{6}$$

where the first term is given by the usual Lagrangian one obtains for a set of charged particles in the presence of a magnetic field

$$\mathcal{L}_\phi = \phi^\dagger \left(i\hbar\partial_t - e(A_0 + a_0) \right) \phi - \frac{1}{2m} \left| \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A} - \frac{e}{c}\mathbf{a} \right) \phi \right|^2 - \frac{1}{2} \int d^2y \delta\rho(x) V(x-y) \delta\rho(y).$$

The interesting part, namely $\mathcal{L}_a$, is the novel term in this theory, and it constitutes the so-called Chern-Simons Lagrangian:

$$\mathcal{L}_a = \frac{e\pi}{2\theta\phi_0} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{7}$$

and all the physical information will thus be contained in the partition function

$$Z[A_\mu] = \int da_\mu d\phi \; e^{i(S_a[a_\mu] + S_a[A\mu + a_\mu])}. \tag{8}$$

We note that this partition function is, in fact, an exact description of the original problem. No approximations have been made up until now. Although Ginzburg-Landau theory usually presuposes some coarse-graining that leads to a mean-field approximate theory (at least as far as I have studied), the theory that comes out of Eq.(8) is called a Chern-Simons-Ginzburg-Landau (CSGL) theory by Zhang and collaborators.

Notwithstanding this question of semantics, they do end up reducing the exact path-integral using a mean-field approximation to obtain the low energy theory that would describe the FQHE. If we integrate out the external gauge field A_μ, we get at the mean field level, the effective theory for the bosonic phase and density degrees of freedom

$$\mathcal{L}_\phi^{eff} = -\delta\rho(\partial_t\theta + e\delta a_0) - \frac{\bar{\rho}}{2m} \left(\nabla\theta - e\delta\mathbf{a} \right)^2 - \frac{1}{2} \delta\rho V(x-y) \delta\rho(y). \tag{9}$$

The low energy theory is then obtained by integrating out the $\delta\rho$ fluctuations. A similar procedure can be carried to obtain an effective Lagrangian for the external field.

In principle, having such an effective Lagrangian at hand, one can go ahead and compute all the correlation functions and, consequently, the response of the system to various external probes. In particular, in this way one obtains explicit expressions for the Hall conductivity, the compressibility, etc., from which we can then try to quantitatively conclude that the theory is consistent with experimental observation.

However, since we want to avoid a technical description, let us instead address the consequences of this Ginzburg-Landau theory in terms of the heuristic physical picture that emerges from this mapping into bosons. We will now try to cast the phenomena observed in the FQHE in the language of familiar to the theme of charged superfluids, following the physical pictures by Zhang [14].

3.2 On the other side of the coin: a charged superfluid

The key heuristic idea that results from the CSGL theory can be understood in terms of flux attachment. From the equations of motion, one can write the relation

$$\nabla \times \mathbf{a}(x) = (2k+1)\phi_0 \rho(x) \hat{z} \tag{10}$$

where the factor $(2k+1)$ comes from the singular gauge transformation restriction we found in the previous section. What does this expression tell us? The left-hand side is the magnetic field associated to the field $\mathbf{a}$. The right hand side constitutes a magnetic field that depends on the *density of bosons.* Thus, it appears to be that the bosons themselves are creating this effective gauge field.

We can then put forth the following interpretation: each original electron is mapped into a boson to which we attach an odd number of flux quanta $\phi_0 = hc/e$. This attachment comes about from the appearance of the $\mathbf{a}$ gauge field we introduced earlier. Since flux is attached to each boson, one would expect then that the associated magnetic field is proportional to the density. Thus we see that the density of magnetic flux is given by $\rho_a = (2k+1)\rho(x)$. Let us then exploit this interpretation that emerges from the CSGL theory in order to describe the FQHE in terms of this novel fluid of charged bosons.

Fractional filling: Since we are dealing with a charged superfluid, we expect there to be a Meissner effect i.e. there should be an exclusion of magnetic field from the bulk of the system, just as what happens in superconductors. This means that somehow the external magnetic field should be made to cancel inside the fluid. The only way, however, that this can happen is if the magnetic flux attached to the bosons is strong enough that it can exactly cancel the external magnetic field. This cancelation will occur when the flux density of both fields exactly match each other. Hence, we must have

$$\rho_a = \rho_A \quad \rightarrow \quad (2k+1)\rho = \frac{1}{\nu}\rho \quad \rightarrow \quad \nu = \frac{1}{2k+1} \tag{11}$$

That is, the last equality tells us that the only way we can get a Meissner effect in these charged bosons is when the filling fraction of the parent electrons is such that the fractional quantum Hall effect is observed.

Incompressibility and energy gap: By the same line of argument, note that changing the density of the system at some point in the fluid leads to the fluid developing extra flux density $\delta\rho_a(x)$ locally, which means that the external field is no longer cancelled. Since we cannot have such a situation, for small enough disturbances, the system naturally tries to avoid density modulations. That is, the system of bosons (and, by our unitary transformation, of fermions) is naturally incompressible. Because of this, we then expect there to be an energy gap in the system, which corresponds to the gap in the FQHE.

Quantum Hall effect: Now let us imagine how current flows when the superfluid is subject to a potential drop. Since each boson still carries charge $q = e$ (this is the original charge of the electron, which has not changed), then the net current in a given direction is $I_q = e\frac{dN}{dt}$, with N the number of electrons.

Note as well that each boson carries magnetic flux as it traverses the sample. As a consquence, using Faraday's law, as the bosons move, they create an emf which is perpendicular to the direction of the current iself. Hence, the induced voltage drop is given by

$$V = \frac{1}{c}\frac{d}{dt}\left[(2k+1)\phi_0 N\right] = (2k+1)\frac{h}{e}\frac{dN}{dt} = (2k+1)\frac{h}{e^2}I_q \tag{12}$$

which means that, in this rough calculation, the Hall conductivity is given by

$$\sigma_{xy} = \frac{1}{2k+1}\frac{e^2}{h} \tag{13}$$

Thus, the Hall conductivity of the charged bosons matches the Hall conductivity measured for the FQH states.

Quasiparticles: fractional charge and statistics We can now consider excitations in the superfluid. In a neutral superfluid, the lowest excited states are phonons, which are gapless. However, in a charged bosonic system, the interactions push these excitations to the plasma frequency, which is a higher energy scale. There are two other excitations in a superfluid, namely rotons and vortices.

Regarding vortex excitations, we can construct vortices which have a finite energy by requiring the vorticity to be quantized $\oint d\mathbf{l} \cdot \delta\mathbf{a} = m\phi_0$, where $\delta\mathbf{a}$ is the field disturbance caused by the vortex excitation. Since this field disturbance is directly related to how the density of bosons is distributed in the system, then there should be associated to each vortex an excess or absence of charge. Using Eq. (10), we can compute the charge of this vortex excitation by noting that

$$\int d\mathbf{A} \cdot \nabla \times \delta\mathbf{a} = \int dA(2k+1)\phi_0\delta\rho \quad \rightarrow \quad \oint d\mathbf{l} \cdot \delta\mathbf{a} = (2k+1)\frac{\phi_0}{e}\delta Q \tag{14}$$

Hence, the extra charge carried by a vortex with $|m| = 1$ is $\delta Q_\pm = \frac{\pm e}{2k+1}$. We can then match these excitations with the quasielectrons and quasiholes in Laughlin's theory, since this is the same amount of fractional charge that has been proven to be carried by excitations in FQH states.

Since the quasiparticles we just mentioned carry fractional charge and unit flux, it follows that the Aharonov-Bohm phase that the many-body wavefunction gains as a result of the exchange of two quasiparticles is $\Delta\phi = \frac{1}{2}\frac{q\phi_0}{\hbar} = \frac{\pi}{2k+1}$. Hence, the total phase gained under exchange is $e^{i\left(\frac{\pi}{2k+1}\right)}$ i.e. the quasiparticles obey fractional statistics.

Ground state wave function: The superfluid theory that emerges is actually able yield far greater information: it is able to provide an approximation to the correlated many-body ground state. How can this be done? Roughly speaking, we average out the effect of the gauge fields, and obtain a theory describing the low energy behaviour of the bosons in the system by keeping the phase fluctuations only. Once this is done, one then realizes that the resulting theory is reduced to the description of a set of harmonic oscillators, for which we know analyticially what the structure of the ground should be.

It is natural then to ask, in what manner does this approximate ground state differ from that provided by Laughlin's wavefunction. The answer: it doesn't. Remarkably, the low energy limit of the CSGL theory leads exactly to Laughlin's wave function. Hence, not only can the theory describe the macroscopic properties of the FQHE, but it can also produce results concerning the microscopics as well. This contrasts with, for example, the Ginzburg-Landau theory of superconductivity. That theory was designed to provide an explanation of the macrascopic phenomena of the superconducting state. It cannot provide information about microscopic mechanisms, the BCS theory being the genuine answer to that problem. In the case of the CSGL theory, however, both worlds, the macroscopic and microscopic, are surprisingly conquered.

From these few simple calculations and physical arguments, one can see the power of the CGSL theory. The charged superfluid language provides another convenient way of understanding the physics of the fractional quantum Hall effect. Let us now discuss what type of correlations we can discover in this superfluid and, in this way, learn something new about the ordering in the FQH states.

3.3 Off-diagonal long-range order gets revealed

Within the mean-field treatment put forth by Zhang, the correlation function $\langle \phi^\dagger(x)\phi(y)\rangle$ can be calculated by integrating out the gauge field and the high-energy density fluctuations in the density of the bosonic field. The effective action can then be written in terms of the phase degree of freedom, which leads to a correlation function of the form

$$\langle \theta(-q)\theta(q)\rangle \approx -\frac{1}{2\nu}\frac{2\pi}{q^2} \quad \rightarrow \quad \langle \phi^\dagger(x)\phi(y)\rangle \sim |x-y|^{-\frac{1}{2\nu}} \tag{15}$$

Hence, we see that, although long-range correlations in the superfluid decay with distance, this decay is algebraic. Hence, we can say that the charged superfluid posseses algebraic off-diagonal long-range order and, by extension, we can say that this correlation is a hidden property of the fractional quantum Hall states [14].

This insight into the type of correlations of the FQH states was also explored from a completely independent perspsective by Girvin [5]. In that study, a singular unitary transformation was performed as well, except that it was done directly on Laughlin's variational wave function. That allowed Girvin to derive the reduced single particle density matrix of the corresponding bosons, which yielded precisely the same exponent as in Eq. (15), which was derived using the charged superfluid language.

Let us better conceptualize the notion that the type of long-range order we are talking about here is in a sense hidden. If we calculate the reduced single particle density matrix

$$\rho(z,z') = \frac{N}{Z}\int d^2z_2 \ldots d^2z_N \psi^*(z,z_2,\ldots,z_N)\psi(z',z_2,\ldots,z_N) \tag{16}$$

using directly Laughlin's wave functions, the corresponding integral would be largely canceled out by the phases in the wave function, and the resulting density matrix would be short ranged. Consequently, it is not explicty from Laughlin's wave function that there is any long-range order at all. However, if we perform the singular transformation, we now obtain

$$\rho(z,z') = \frac{N}{Z}\int d^2z_2 \ldots d^2z_N \; e^{-i\frac{e}{\hbar c}\int_z^{z'} d\vec{r}\cdot\vec{A}}\psi^*(z,z_2,\ldots,z_N)\psi(z',z_2,\ldots,z_N) \tag{17}$$

The additional phase factor that results from the transformation turns out to cancel precisely the phases of Laughlin's wave functions. This then unveils an underlying long-range order which, in Zhang's treatment, is a long-range order in the associated two-dimensional superfluid.

In view of all of this, we can ask whether this type of order is necessarily linked to the existence of the FQHE. In other words, does the presence of algebraic order in the superfluid relate to the existence of the long wavelength excitation gap that stabilizes the FQH states? The answer to this is in the positive, and it has to do with the binding of the electrons to the holes in Lauglin's the wave function. Let us thus find this important connection, namely between the superfluid picture and Laughlin's microscopic theory.

In general, because of the presence of the magnetic field, the wave function of the many-body system will have zeroes, as we argued before. Each flux quantum depletes the wave function at the position at which it is piercing the sample. Since fermionic wave functions already have vanishing points due to their antisymmetry properties, then the energetically optimal configuration is to let the fermions themselves move towards the zeroes of ψ. In this way, there's no need to create extra zeros in the ground state i.e. new vortex configurations, which end up costing a finite amount of energy.

Because of this reasoning, one can see that the appearance of the algebraic long-range order in the superfluid system will occur, *if and only if* the parent fermions condense into the zeroes of the many-body wave function thus forming a FQH state. Hence, the emergence of such order in the superfluid picture is, indeed, indicative of the appearance of the FQHE in the fermionic system. The Chern-Simons-Ginzubrg-Landau theory serves to enveil this deep connection between the hidden algebraic correlation in the superfluid and the existence of the fractional quantum Hall state.

4 Conclusion and today's take

We have presented here a few qualitative aspects of the CSGL theory. It provides us with a way to understand the fractional quantum Hall effect in terms of superfluidity. As we mentioned at the beginning, studying a system from various perspectives provides a coherent idea of the fundamental physics at work. The treatment put forth by the CSGL theory achieves this unifying goal.

Let us note, however, that this type of study has been challenged by a different point of view. As Wen has pointed out [13], the approach presented here does not actually address the fundamental aspects of the fractional quantum Hall effect. Even though this theory seems to describe the relevant phenomena of these states, there are some cases in which this description fails. As it turns out, the correct description goes beyond the usual Landau-type description of local order parameters.

Instead, in the fractional quantum Hall effect, as argued by Wen, the physics at work intrinsically in these states cannot be completely characterized by off-diagonal long-range orders. Instead, one has to make use of the concepts of the so-called topological orders and the asociated topological field theories. This is an interesting topic which is well-worth pursuing as a continuation of what we have learned in writing this essay.

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