

University of Illinois, Urbana-Champaign

---

Yulia Maximenko

## **Superconducting phase transition**

Phys 569 ESM Term Essay

December 15, 2011

# Contents

<b>1</b>	<b>Review</b>	<b>1</b>
1.1	Ginzburg-Landau theory . . . . .	1
1.2	Types of superconductivity . . . . .	2
1.3	Vortex interaction in type-II superconductors . . . . .	3
<b>2</b>	<b>Phase transition kinetics</b>	<b>4</b>
2.1	Time-dependent Ginzburg-Landau functional . . . . .	4
2.2	Computer simulations . . . . .	6
<b>3</b>	<b>Technological applications</b>	<b>8</b>
<b>4</b>	<b>Conclusion</b>	<b>10</b>

## 1 Review

Phase transitions occur in the great variety of physical systems. The first order phase transitions are well studied, and humanity has been dealing with them from literally primeval times. On the other hand, the second order phase transitions are not that straightforward: they have been concealed from keen scientists for a really long time. To name some of them, we can think of paramagnet-ferromagnet, fluid-superfluid, amorphous-crystal and normal-superconducting transitions, etc. Such a wide range of branches did not prevent from creating a general theory of phase transitions, which can be applied with certain limitations to any second order phase transition. Here we will try to study superconducting-normal phase transition for both types of superconductors, using mostly Landau theory of second order phase transition, or particularly Ginzburg-Landau (GL) functional for superconductivity. A review on the time-dependent GL theory, numerical simulations and problems arising will be reported. One possible technological application in accelerator physics will be discussed as well.

### 1.1 Ginzburg-Landau theory

V. L. Ginzburg and L. D. Landau were the first who combined order parameter and wave function to describe superconducting phase transition. In Landau theory the free energy is expanded in powers of the order parameter.  $|\Psi(\mathbf{r})|^2$  is taken to equal  $n_s/2$ , where  $n_s$  is the superconducting electron density. In the presence of magnetic field, the Gibbs free energy is as follows [1]:

$$\mathcal{G}_{sH} = \mathcal{G}_n + \int \left[ \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{4m} \left| -i\hbar \nabla \Psi - \frac{2e}{c} \mathbf{A} \Psi \right|^2 + \frac{(\text{rot } \mathbf{A})^2}{4\pi} - \frac{(\text{rot } \mathbf{A}) \cdot \mathbf{H}_0}{4\pi} \right] dV, \quad (1)$$

To find the solution one needs to minimize the Gibbs free energy, i.e. to solve the variational problem with respect to  $\Psi^*(\mathbf{r})$ ,  $\Psi(\mathbf{r})$  and  $\mathbf{A}(\mathbf{r})$ . This operation leads to the stationary Ginzburg-Landau equations, providing the variational derivatives are zero. Two parameters are introduced in GL-theory:

$$\xi^2 = \frac{\hbar^2}{4m|\alpha|} \quad (2)$$

is the coherence length, which is essentially the scale of order parameter variation, and

$$\lambda^2 = \frac{mc^2}{4\pi n_s e^2} = \frac{mc^2 \beta}{8\pi e^2 |\alpha|} \quad (3)$$

is the London penetration depth, characterizing the magnetic field decay in a superconductor.

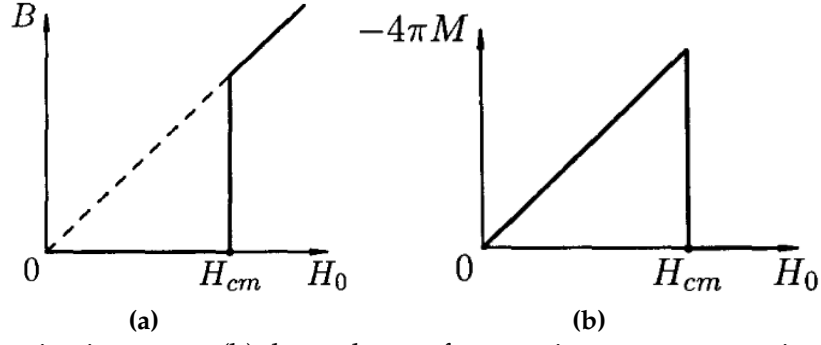
The Ginzburg-Landau parameter

$$\kappa = \lambda / \xi \quad (4)$$

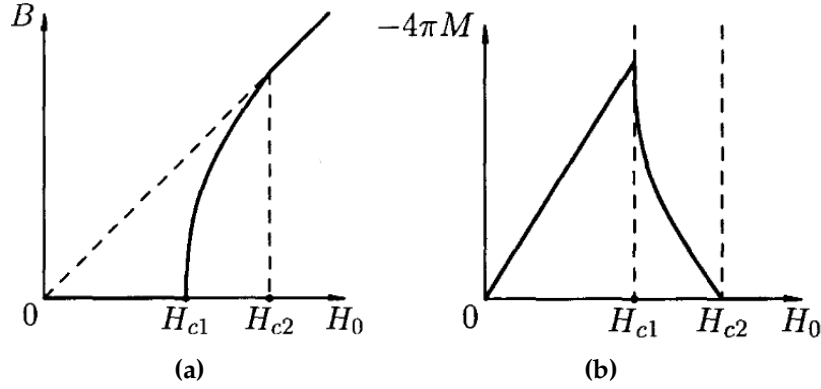
depends on the material and defines the type of the superconductor. For type-I material  $\kappa < 1/\sqrt{2}$  and the surface energy is greater than zero, so it is energetically favorable to form continuous superconducting or normal state. For type-II  $\kappa > 1/\sqrt{2}$  and the surface energy is less than zero, so it is favorable to form finely intermixed composite state of normal and superconducting phase at some range of applied field.

## 1.2 Types of superconductivity

Type-I superconductivity was discovered first and for more than 40 years the different type existence was not being suspected. A. A. Abrikosov in his fundamental work [2] predicted another type of superconductivity, giving reason to search for the whole new class of phenomena. Type-I superconductors exhibit Meissner-Ochsenfeld effect: they completely eject magnetic field till it reaches some critical value. Then superconductor becomes normal conductor. Different situation takes place in type-II superconductors: magnetic field there starts to penetrate inside the superconducting material if applied field is greater than some critical value  $H_{c1}$ . Unlike for type-I (Fig. 1), for type-II superconductors magnetic field does not completely destroy superconducting phase until it reaches another critical value  $H_{c2}$  (Fig. 2).



**Figure 1.** (a) Magnetization curve, (b) dependence of magnetic moment per unit volume  $M$  on the external field  $H_0$  for type-I superconductor [1].



**Figure 2.** (a) Magnetization curve, (b) dependence of magnetic moment per unit volume  $M$  on the external field  $H_0$  for type-II superconductor [1].

### 1.3 Vortex interaction in type-II superconductors

Magnetic field penetrates the type-II superconductor in the shape of quantum vortex lines. Every vortex line has (a) a normal core along the direction of applied magnetic field, (b) hence a zero order parameter in the center, and (c) a radius, approximately equal to the coherence length  $\xi$ . Vortex current exists in the area of radius  $\lambda$  around the normal core, which in the same time is the approximate distance between closest vortices. If applied magnetic field is increased further,  $H_0 > H_{c1}$ , vortices starts to approach each other, till their normal cores overlap at  $H_{c2}$ . In this case second order phase transition to the normal state occurs.

For type-II superconductors, if  $\lambda \gg \xi$ , the energy of one vortex line is given by [3]

$$F = \left( \frac{\Phi_0}{4\pi\lambda} \right)^2 \cdot \left( \ln \frac{\lambda}{\xi} + \varepsilon \right), \varepsilon \approx 0.1, \quad (5)$$

where  $\Phi_0 = \frac{hc}{2e}$  is flux quantum and  $\varepsilon$  corresponds to the approximately calculated contribution of the normal core.

Energy of two interacting vortices [3]

$$F = F_1 + F_2 + U_{12} = F_1 + F_2 + \frac{\Phi_0 \cdot h_{12}}{4\pi} L, \quad (6)$$

where  $F_1$  and  $F_2$  are the energies of the isolated vortices,  $L$  is the length of one of the vortex lines,  $h_{12} = h_1(\mathbf{r}_2) = h_2(\mathbf{r}_1) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{\lambda}\right)$  is the field of one vortex line in the center of another vortex line,  $K_0$  is the zero-order Bessel function with imaginary argument. Interaction between two vortices in type-II superconductors is thereby repulsive. Normally vortices form a regular triangle lattice [1].

Vortex approach can also be used in type-I superconductors in order to demonstrate the mechanism of the first order phase transition. Surface energy for  $\kappa > \frac{1}{\sqrt{2}}$  is positive, vortex interaction is attractive, so vortices tend to form continuum medium, extirpating as far as possible the normal-superconducting interface.

## 2 Phase transition kinetics

### 2.1 Time-dependent Ginzburg-Landau functional

Previous speculations about Ginzburg-Landau functional concerned only stationary order parameter (1). As long as we would like to study the time-dependent phase transition properties, it is natural to assume that one should use Ginzburg-Landau theory as a good technique for phase transition description. Not only spatial variation of the order parameter should be found, but also time variation. Non-stationary Ginzburg-Landau theory generalization would do perfectly in this case, and one of the most straightforward applications would be the normal-superconducting interface propagation. The problem of time-dependent theory of superconductivity drew attention for many years, and a derivation of the generalized time-dependent GL-theory without a hitch was an obsession [4]. Temperature range simplifications made in stationary GL-theory gave hope for the possibility of the same simplifications in time-dependent theory of superconductivity. Unfortunately, it turned out to be in general untrue. The dynamic behavior of the order parameter cannot be described by a simple differential equation [4]. Nonlinear terms can only be derived in case of gapless superconductor. The complication follows the singularity in the density of states (DOS) on the edge of the superconducting energy gap [5][6]. This singularity results in the appearance of the weakly damping, oscillating terms, i.e. the dissipative processes occur in the superconductor due to the time variation of the order parameter. Earlier works were either wrong, or restricted to the linear term in GL functional or to some other more grave conditions, than it was implied by the authors. Gor'kov and Eliashberg [5] were the first who derived the only strict time-dependent version of GL equation, applicable in a reasonable range of fields and frequencies, but still valid only for a gapless superconductor. Magnetic impurities and other decoupling mechanisms weaken the

DOS singularity in BCS [7], and the energy spectrum becomes gapless. In this case the energy can be expanded in powers of  $\Delta/\alpha$  and  $\omega/\alpha$ , where  $\alpha$  is the decoupling parameter [7]. This approach was used by Gor'kov and Eliashberg.

From now we will assume the superconductor to be always in the state of local equilibrium with the charge density of the electrons  $\rho$ , internal energy  $E_{int}$ , vector potential  $\mathbf{A}$  and order parameter  $\delta$  [4]. Conservation laws for  $\rho$ , entropy  $S$  and total energy  $E$  are valid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (7)$$

$$\frac{\partial S}{\partial t} + \nabla \cdot \mathbf{J}^S = \frac{R}{T}, \quad (8)$$

$$\frac{\partial E_t}{\partial t} + \nabla \cdot \mathbf{J}^E = 0, \quad (9)$$

where  $\mathbf{J}$ ,  $\mathbf{J}^S$ , and  $\mathbf{J}^E$  are the currents of respectively charge, entropy and energy,  $R$  is the dissipative function. Besides the losses due to Joule heating, there are also losses as a result of finite relaxation time of the superconducting electrons, according to Tinkham (1964). In order to describe the superconductor, the Maxwell equations are also used, as well as the "equation of motion" for the order parameter. As it usually done in phenomenological approach, we will take the order parameter time derivative to be equal with some constant to the functional derivative of the free energy with respect to order parameter. Free energy functional is acting like a generalized force:

$$\gamma \frac{\partial \Delta}{\partial t} = - \frac{\partial F}{\partial \Delta^*}, \quad \gamma > 0. \quad (10)$$

This equation is not gauge invariant, so we should replace it with the corrected one:

$$\gamma \left( \frac{\partial \Delta}{\partial t} - 2ie \left( \phi + \frac{\mu}{e} \right) \right) = - \frac{\partial F}{\partial \Delta^*}, \quad (11)$$

where  $\phi$  is the scalar potential and  $\mu$  is the chemical potential.

Dissipation function  $R$ , derived in [4], includes the terms due to the time variation of the order parameter. We should note here that the Joule heating, caused by the energy dissipation of changing currents and fields, is neglected.

One of the possible forms for GL time-dependent equations, including Maxwell equations, is as follows [7]:

$$D^{-1} \left( \frac{\partial}{\partial t} + i \frac{2e\psi}{\hbar} \right) \Delta + \frac{1}{\xi^2} (|\Delta|^2 - 1) \Delta + \left( \frac{\nabla}{i} - \frac{2e}{\hbar c} \mathbf{A} \right)^2 \Delta = 0, \quad (12)$$

$$J = \sigma \left( -\nabla\psi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) + \text{Re} \left[ \Delta^* \left( \frac{\nabla}{i} - \frac{2e}{\hbar c} \mathbf{A} \right) \Delta \right] \frac{1}{8\pi\lambda^2}, \quad (13)$$

$$\rho = \frac{(\psi - \phi)}{4\pi\lambda_{TF}^2}, \quad (14)$$

where  $D$  is the diffusion constant in normal state;  $\psi$  is the chemical potential divided by electron charge;  $\Delta$  is the dimensionless gap, normalized by  $\Delta_0 = \pi k \sqrt{(2(T_c^2 - T^2))}$ ,  $\lambda_{TF}$  is the Thomas-Fermi static charge screening length. Thomas-Fermi screening approximation requires the slow total potential variation, the constant chemical potential and the low temperature. The coherence length here is  $\xi = \hbar(6D/\tau_s)$ , where  $\tau_s$  is the spin-flip scattering time. The penetration depth is  $\lambda = \hbar c(8\pi\sigma\tau_s)^{-1/2}/\Delta_0$ .

Since Gor'kov and Eliashberg, as far as I understand, no significant progress in simplification of the valid time-dependent GL equation (TDGL) was made. Further research either dealt with the equations in the form similar to that of (12)-(14), or tried to improve the accuracy by adding nonlinear terms, delving into the microscopic Bogoliubov–de Gennes theory and Green function apparatus (e.g. [4], [8], [9], etc).

Solving the discussed equations for a more or less real situation seemed extremely difficult or sometimes impossible, but with the progress of computer simulations the abundance of opportunities arose. In the next section we will try to review some of the most interesting and related to the realistic processes numerical experiments, carried out in the last twenty years.

## 2.2 Computer simulations

During last twenty years, papers on the simulated superconducting-normal or normal-superconducting phase transition have kept being published. Interest in this topic does not seem to drop. Computer performance keeps increasing exponentially (Moore's Law), and new problems appear possible to solve, which could not be solved before.

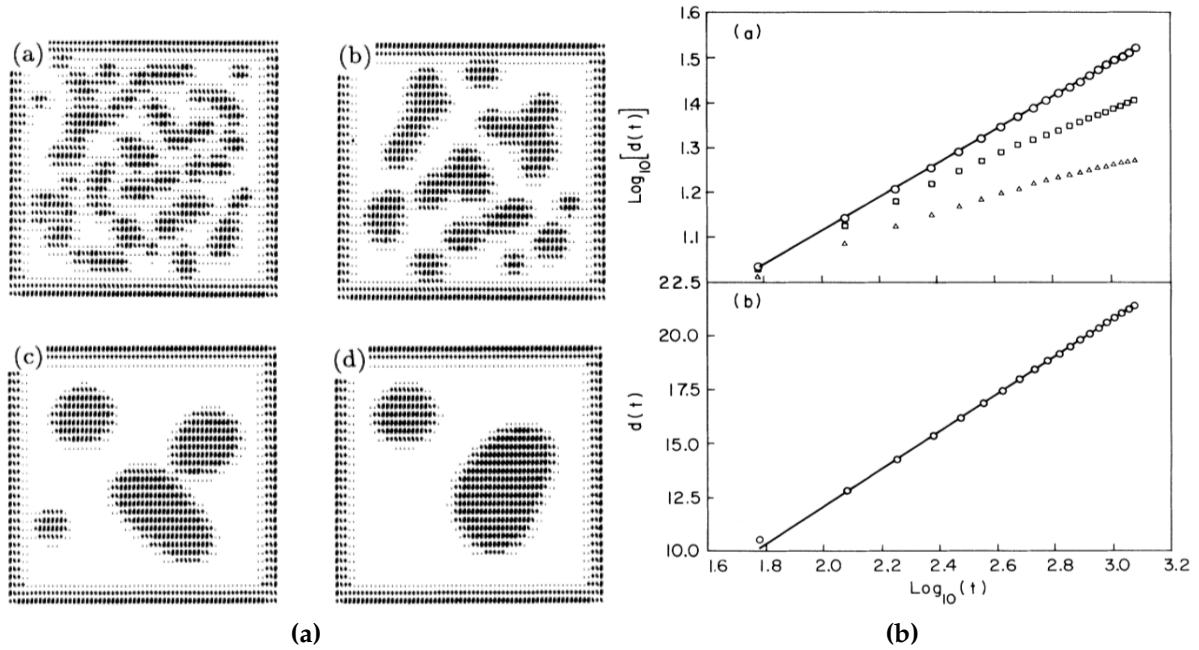
In the 1990 year paper [10] by F. Liu, M. Mondello and N. Goldenfeld, the 2D TDGL equations are written in a form of

$$\gamma \frac{\partial \psi}{\partial t} = - \left[ a + b|\psi|^2 + \frac{1}{4m} \left( i\hbar \nabla + \frac{2e}{c} \mathbf{A} \right)^2 \right] \psi \quad (15)$$

and

$$\frac{4\pi\sigma}{c^2} \frac{\partial \mathbf{A}}{\partial t} = -\nabla \times (\nabla \times \mathbf{A}) - \frac{2\pi e}{mc} \left[ \psi^* \left( i\hbar \nabla + \frac{2e}{c} \mathbf{A} \right) \psi + c.c \right], \quad (16)$$

where  $\sigma$  is the normal state conductivity. The imposed calibration is  $\nabla \cdot \mathbf{A} = 0$ .



**Figure 3.** (a) Meissner state growth in time in type-I sc at  $T=0$ . (b) Intervortex spacing for various  $G = 1/(2\kappa)$  in zero field for type-II sc. [10]

The dynamics of Meissner phase growth is modeled in a type-I superconductor (Fig. 3 (a)). In type-II superconductor at zero-field the scaling law for the average intervortex spacing  $d$  was found to be violated (Fig. 3 (b)) in some range of Ginzburg-Landau parameter  $\kappa$  and to tend to logarithmic dependence. In this paper it is admitted that TDGL equation are valid only for gapless superconductors near  $T_c$ , but it is assumed that they nevertheless give a “semiquantitative description of the phase-transition kinetics”.

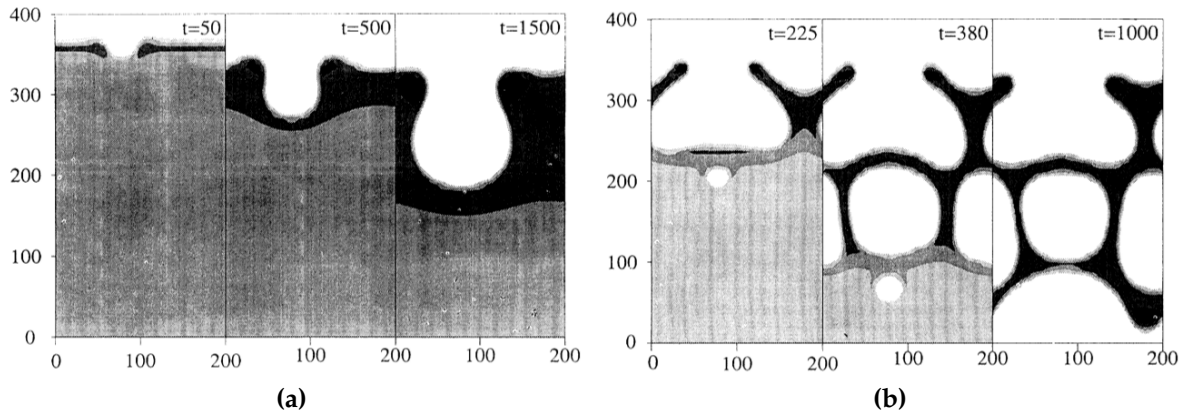
In the same year paper [11] by H. Frahm, S. Ullah and A. Dorsey, the results close to the previous article are reported. The modeled bulbous superconducting front dynamics is shown in Fig. 4 (a) for nucleation regime of external field: only seeds of radius greater than a given critical one can grow. Simulation for spinodal regime with unstable normal phase is shown in Fig. 4 (b). Both figures describe type-I superconductor. The authors observe the analogy to the solid-liquid system.

For type-II superconductor the authors begin with a mixed state regime and a superconducting seed in a uniform field. Interface grows until one vortex penetration into the superconducting phase becomes favorable (Fig. 5). After that the slow triangular lattice formation is demonstrated.

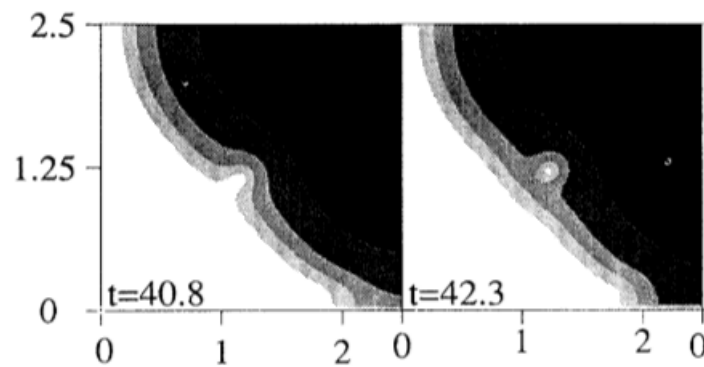
One more interesting study of spatial patterns of order parameter and supercurrent in type-II superconductor was reported in [12] only a year later.

Among modern publications we can mention the following: [13], [14] and [15]. They confirm the persisting interest in this topic.

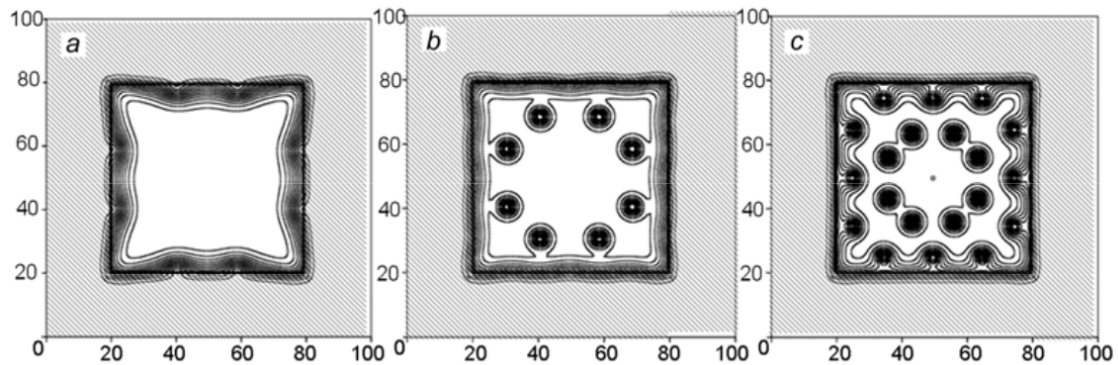




**Figure 4.** Type-I superconductor. (a) Nucleation regime, perturbed surface growth. (b) Spinodal regime with a leaking order parameter through the flux wall. [11]



**Figure 5.** One vortex penetration into the type-II superconductor [11].

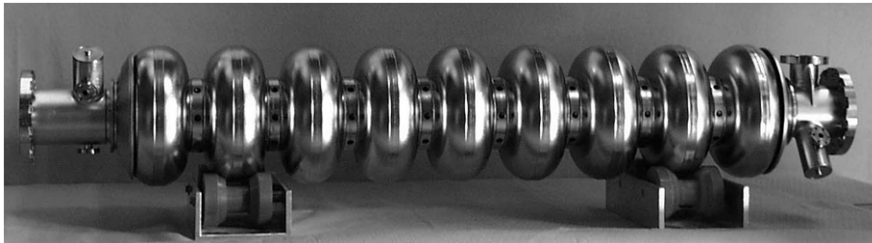


**Figure 6.** Penetration of the vortex structure into the superconductor sample with  $\kappa = 2$  [14].

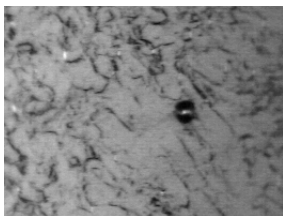
### 3 Technological applications

Superconducting phase transition dynamics research suddenly finds itself necessary in accelerator physics. Superconducting electromagnets are installed along the beam line of circular accelerators in order to bend the beam path. Superconducting radio frequency cavities (Fig. 7) are used to accelerate light particles. In both cases superconducting materials use is justified by lesser energy losses. The current amplitudes in magnets are so high, that in case the magnet becomes normal conductor, the Joule

heating can destroy the whole magnet or at least damage some of its crucial parts. In SRF cavities the effect of superconducting-normal transition is not that dramatic, but still the temperature abruptly increases, helium in the surrounding dewar is boiling, the quenching cavities quality factor drops to zero, causing the loss of all the energy stored in the resonator and hence the zeroing of the accelerating gradient.



**Figure 7.** 1.3 GHz SRF TESLA cavity, Fermilab VTS facility



**Figure 8.** Typical cat-eye defect, Fermilab VTS facility.

In magnets quenching can be controlled by monitoring the current and heat distribution in the system and protected by different means [16], but still quench propagation is of interest there. In SRF technology [17] it becomes crucial, because one has to locate quench origin on the internal surface of resonator. Usually some major defect in the lattice or impurity (Fig. 8) causes the quench, and it is possible to improve cavity performance later by some additional processing. So, quench propagation characteristics can be of significant help to develop a reliable

technique of quench location. Essential parameters in this case are normal zone radius, temperature dependence, time constant of the process, critical values of temperatures and oscillating electromagnetic fields inside the cavity. The whole process is complicated by nontrivial thermal conductivity, resistance, and heat capacity dependence on temperature of real materials, but some straightforward attempts of modeling such a process were made [18][19] for niobium cavities. No attempts to theoretically analyze and predict the characteristic parameters were made at all.

Experimental data [20] claim the temperature of the quench epicenter in Nb cavities to be around 100K, when working temperature range is 1.6-2K. Niobium critical temperature is 9.2K. There exist some experimental techniques of quench locating, based on the temperature or second sound measurements (e.g. [21], [22], [23]), but they are far from being reliable and universal.

The problem here, as I see it, is in certain experimental complications, of course, but also in the lack of interest in condensed matter theory in high energy physicists and vice versa. Maybe, a simple model will actually do for the quench propagation dynamics, but since nobody reports their achievements in this area, the simple model apparently does not describe the process well enough. So, a more accurate model is asked for. Niobium is a type-II superconductor and operating fields are close to the critical value [24], so vortex state may influence the macroscopic parameters. The

combination of time-dependent GL-theory and experimentally measured parameters of Nb could shed some light on the quench process in SRF cavities. From the other hand, there are a lot of complications in time-dependent GL equation itself concerning application temperature range, gapless superconductors, magnetic impurities and oscillating field. Anyway, the topic is important for accelerator physics, and, it is hoped, will draw more attention to itself in the nearest future.

## 4 Conclusion

The time-dependent simulations in superconductivity is undoubtedly a promising and fruitful field in spite of its development for more than 70 years. The same can be said about superconducting phase transitions in particular. Though there is much uncertainty in time-dependent Ginzburg-Landau theory and still there is no strict solution (and hardly there will ever be), the relatively simple case of TDGL (12-14), introduced by Gork'ov and Eliashberg, still matters and was proven to be valid in a quite wide range of temperatures, fields and frequencies, at least semiquantitatively. Nowadays a lot of papers is being published about newly calculated models based on high-performance computing. The TDGL equations are confirmed in asymptotic behavior by classical textbook superconductivity phenomena, which once more confirms their relative validity.

As it was shown, time-dependent superconducting phase transition simulation could be of great help not only in fundamental condensed matter physics, but also in high energy physics. Superconducting quench processes in SRF cavities as well as in superconducting magnets definitely require some qualified research in order to improve safety, eliminate excessive expenses and increase efficiency of particle accelerators.

Possible further directions of research could be the time-dependent study of order parameter and supercurrent patterns in phase transition in superconductors with intermediate  $\kappa = \frac{1}{\sqrt{2}}$ . Also in the light of recent discovery of so called type 1.5 superconductivity, it would be exciting to model pattern formation in magnesium diboride and in two-band superconductors in general. Though phase transition does not seem to produce any uncommon vortex structures, TDGL can still be used to model the behavior of the MgB<sub>2</sub> in the slightly extended range.

## References

- [1] V.V. Shmidt. *Introduction to the physics of superconductors*. MCNMO, 2009.
- [2] A. A. Abrikosov. On the magnetic properties of superconductors of the second group. *Sov. Phys. JETP*, 5:1174–1182, 1957.
- [3] P. G. De Gennes. *Superconductivity of metals and alloys*. MIR, 1968.

- [4] M. Cyrot. Ginzburg-landau theory for superconductors. *Rep. Prog. Phys.*, 36:103, 1973.
- [5] L. P. Gor'kov and G. M. Eliashberg. Generalization of the ginzburg-landau equations for non-stationary problems in the case of alloys with paramagnetic impurities. *JETP*, 54:612, 1968.
- [6] G. M. Eliashberg. Nonstationary equations for superconductors with low concentration of paramagnetic impurities. *Sov. Phys. JETP*, 28:1298, 1969.
- [7] M. Tinkham. *Introduction to superconductivity*. Moscow, Atomizdat, 1980.
- [8] K. Binder. Time-dependent ginzburg-landau theory of nonequilibrium relaxation. *Phys. Rev. B*, 8:3423–3438, Oct 1973.
- [9] T. J. Rieger, D. J. Scalapino, and J. E. Mercereau. Time-dependent superconductivity and quantum dissipation. *Phys. Rev. B*, 6:1734–1743, Sep 1972.
- [10] Fong Liu, M. Mondello, and Nigel Goldenfeld. Kinetics of the superconducting transition. *Phys. Rev. Lett.*, 66:3071–3074, Jun 1991.
- [11] Holger Frahm, Salman Ullah, and Alan T. Dorsey. Flux dynamics and the growth of the superconducting phase. *Phys. Rev. Lett.*, 66:3067–3070, Jun 1991.
- [12] Y. Enomoto and R. Kato. Computer simulation of a two-dimensional type-ii superconductor in a magnetic field. *J. Phys.:Condens. Matter*, 3:375–380, 1991.
- [13] Nicholas Guttenberg and Nigel Goldenfeld. Ordering dynamics in type-ii superconductors. *Phys. Rev. E*, 74:066202, Dec 2006.
- [14] K. S. Grishakov et al. Modeling of superconductors based on the time-dependent ginzburg-landau equations. *Russian Physics Journal*, 52-11, 2009.
- [15] A. Bulgac et al. Real-time dynamics of quantized vortices in a unitary fermi superfluid. *Science* 10, 332 no. 6035 pp.:1288–1291, 2011.
- [16] J. H. Schultz. *Superconducting magnets, quench protection*. Wiley encyclopedia of electrical and electronics engineering., 1999.
- [17] B.Aune and etc. Superconducting tesla cavities. *Physical Review Special Topics - Accelerators and Beams*, 3(2), 2000.
- [18] S H Kim. Simulation of quench dynamics in srf cavities under pulsed operation. In *20th IEEE Particle Accelerator Conference, Portland, OR, USA*, 2003.
- [19] Yu. Maximenko et al. Quench dynamics in srf cavities. In *SRF 2011, Chicago, IL, USA*, 2011.
- [20] J. Ozelis et al. Design and commissioning of fermilab's vertical test stand for ilc srf cavities. In *PAC07, Albuquerque, WEPMN106*, 2007.
- [21] J.Knobloch. *Advanced thermometry studies of superconducting radio-frequency cavities*. PhD thesis, Cornell University, 1997.

- [22] G.Ciovati. *Investigation of the superconducting properties of niobium radio-frequency cavities*. PhD thesis, Old Dominion University, 2005.
- [23] H.Padamsee. *RF superconductivity*. WILEY-VCH, 2009.
- [24] N. R. A. Valles and M. U. Liepe. Temperature dependence of the superheating field in niobium. *arXiv*, 1002.3182v1, 2010.