Self organization of a Bose Einstein Condensate in an optical cavity

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Abstract

Here, we discuss the spatial self organization of a BEC in a single mode optical cavity driven transversely by a laser field. Above a critical pump power, the atoms organize themselves into one of two possible arrangements depending on the initial fluctuations of the system. The system is analyzed at a mean field level. Connections of this system to the Dicke quantum phase transition is drawn. It is realized by two photon processes involving the cavity field and the pump mode. Finally some experimental results are also reported.

1 Introduction

Recent developments in ultracold atomic gases have helped simulate a range of condensed matter phenomenon,for example the transition between superfluid and Mott insulator states in a Bose Gas. However, the systems realized so far involve the creation of a static optical lattice which is formed by a system of counterpropagating laser beams. The electric field induces atomic dipoles which get trapped in the valleys of the optical potential. However, the imposition of an external lattice prevents us from studying phenomenon which arise from the emergence of a lattice spontaneously because of the interactions. Thus, interesting phenomenon like dynamics of glassy media, crystallization, defects have remained inaccessible to cold atom experiments.

That is why it is an interesting problem to see how an emergent lattice can be obtained in experiments as a result of interactions between the gauge field(photons) and the matter field (which can involve either a gas of coherently driven dipolar atoms or a Bose Einstein gas trapped in a cavity oscillator). The study of these systems is interesting in their own right because they draw elements from cavity QED where we see how the motional degrees of freedom of a mechanical system are affected because of radiation pressure force due to the exchange of momentum between light and matter as well as from the field of ultracold atomic gases.

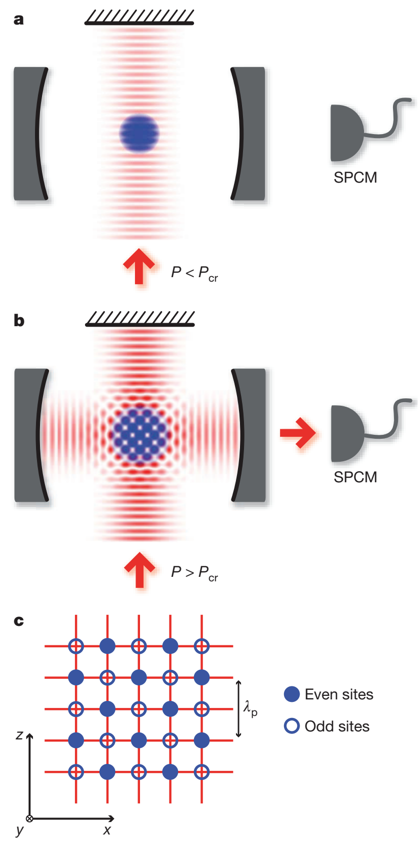

Figure 1: Figure from [1], SPCM-single photon counting module, a) Pump less that critical power, homogenous state, no coherent scattering of light into the detector. b) and c). Self organization into odd and even sites formed by the cavity mode and the pump laser.The pump laser is along the z) axis and the cavity axis is along x direction. Now here is superradiant scattering into the counting module

1.1 Description of the basic setup

A diagram of the basic setup is shown in Figure 1. The setup consists of a Bose Einstein condensate interacting with a single mode of a high Q optical cavity. The condensate atoms are driven transversely from the side by a pump laser mode with frequency ω. The laser is far detuned from the atomic transition ω_A. Thus we assume that $\omega << \omega_A$. Let $|\Delta_A| = |\omega - \omega_A| >> \gamma$, the width of the atomic transition. Thus the excited states of the atoms can be neglected completely. This is nice because we realize that the setup does not require the atoms to satisfy some sort of specific internal configuration.

However, we want the pump laser mode to be resonant to the cavity frequency ω_C, so that the incident pump photons are efficiently scattered into the cavity. Thus, $\Delta_c = |\omega - \omega_C| \simeq \kappa$, the cavity mode linewidth. The essential ingredient of such schemes is that dissipation takes place via cavity decay and not by means of spontaneous emission. The coupling strength between the atoms and the photons are given by the single photon Rabi frequency g which is in the range of κ. Here, g describes the coupling between atoms and the mode. Feeding the cavity by atom laser scattering appears as an effective pump with strength $\eta = \Omega g/\Delta_A$, which can be thought of as a two photon Rabi Frequency.. where Ω is Rabi frequency of the assumedly homogenous transverse laser field perpendicular to the axis. What we show here is that as the pumping strength is increased beyond a certain critical value given by $\eta_{critical}$ the atoms of the Bose Einstein condensate self organize at either the even or the odd antinodes of the system.

1.2 The intuitive picture

Before going into the details of the calculations let us try to understand the intuitive picture. **The discussion here is drawn from [2]**. We, must realize that that we can no longer think in terms of the single particle picture, all the atoms in the cavity are coupled to the same field modes as dissipation channels and this leads to effective long range atom interactions. The atoms are pumped by the transverse field modes and they coherently scatter photons into the cavity. The atoms are in the standing mode wavefunction $\cos(kx)$ of the cavity. The atoms at the nodes do not scatter whereas the maximum scattering happens from the atoms localized near the antinodes of the system. The scattered field amplitude for an atom depends on the position of the atom inside the cavity. For a uniform field distribution, atoms separated by half a wavelength contribute with opposite phase and hence they cancel each other out due to destructive interference. So there is no coherent buildup of field inside the cavity.

However, there are small density fluctuations in any system, hence there is a small buildup of field. For, $\Delta_A < 0$, this leads to an attractive potential towards the antinodes of the cavity, thus inducing periodic localization of the atoms. Thus we have a periodic array of atoms which enhance scattering in certain specific directions, akin to Bragg scattering. If they coherently scatter in the direction of the optical cavity, then this further deepens the periodic potential leading to more atoms being localized, enhanced Bragg scattering and thus we have a feedback loop which further stabilizes the self organization of the system into a periodic pattern. This process is of course accompanied by dissipation and loss from the cavity until the system reaches the steady state. But, all is not well, for the atoms which are now separated by $\lambda/2$ should radiate out of phase with each other thus effectively cancelling each other out and thus no process as described till now should occur. This we will examine from the quantitative point of view. But basically, what happens that there is a spontaneous breaking of symmetry driven by fluctuations which leads the atoms to congregate either in the 'even' or 'odd' antinodes thus leading to constructive interference. The final state is thus characterized by a breaking of translational symmetry and crystallization, all the atoms are either in the even position where $\cos(kx) = 1$ or odd position where $\cos(kx) = -1$. It turns out that this happens only at a field intensity of the lasing mode which is greater than a critical value.

The other interesting aspect of this phase transition to the ordered state is that constructive interference between the various scatterers leads to the buildup of a scattered field intensity which is quadratic in N, the no. of atoms, as happens when all the field amplitudes add up. This happens to be the same as the Dicke phase transition. What happens in that case is that we have a system of interacting two level atoms where each of them are coupled to an electromagnetic field mode. For sufficient coupling(in our case given critical pump strength) the system enters the superradiant phase with macroscopic occupation of the field mode. We later demonstrate how this system is related to the Dicke model.

There have been experiments with BEC's trapped in an atomic cavity and driven by a transverse laser mode that have shown the onset of self-organization and superradiance to be equivalent. In our case the two level system corresponds to two different momentum states that are coupled via the field mode. At self organization the system moves from a flat superfluid to a quantum phase characterized by the macroscopic occupation of the higher order momentum mode and cavity mode as well as spontaneous symmetry breaking both in the atomic density as well as relative phase between the pump and cavity field. The presence of crystalline order along with off diagonal long range order allows us to regard this phase as a supersolid.

2 The Model

This whole section is drawn from [7]. To understand this phase transition let us look at the dynamics in one dimension x along the cavity mode. This is a good approximation if the transverse size of the cavity is much bigger when compared with the waist of the condensate, trapping being achieved for example by means of a transverse magnetic trap. The cavity field is described by the complex amplitude α where $|\alpha|$ is the square root of the photon number. The system of equations governing the motion of the cavity mode and the condensate is analyzed here at the mean field level.

$$i\frac{\partial}{\partial t}\alpha = [-\Delta_C + N\langle U(x)\rangle - i\kappa]\alpha + N\langle \eta_t(x)\rangle \tag{1a}$$

$$i\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{p^2}{2\hbar m} + |\alpha(t)|^2 U(x) + 2Re\{\alpha(t)\}\eta_t(x) + Ng_c|\psi(x,t)|^2\right)\psi(x,t) \tag{1b}$$

Here we are not going to derive these equations, they can be found in [4],[6], but let us talk about them for a moment. Let us, look at 1a. Each atom changes the cavity resonance frequency is a spatially dependent manner, $U = U_0\cos^2(kx)$ where $U_0 = g^2/\Delta_A$ is the amplitude of the shift. As, remarked earlier, for us $U_0 < 0$, in which case the atoms go to the antinodes of the cavity. Since we are at the mean field level we average over the single atom wave function ψ and then multiply by N to get the total shift. Here, κ denotes the loss from the cavity. The factor of Δ_C comes when we go to the frame rotating at the laser frequency ω. The pump term due to scattering between the lasing mode and the cavity mode η has the spatial dependence of the mode function,$\eta \cos(kx)$ and we average this too over the single particle wave function and multiply by N. The equation of evolution of the condensate 1b is a Gross-Pitaevskii equation. The second term on the left hand side is the backaction of the light shift and hence is proportion to U and the photon number $|\alpha|^2$. The third term can be interpreted as the back action of the scattering between the pump mode and the cavity mode. The $2Re\{\alpha(t)\}$ term arises due to $\alpha + \alpha^\dagger$, which corresponds to the absorption or creation of a cavity photon respectively mediated by the atoms. The last term is present due to the s-wave scattering between the atoms and is not due to interaction with any radiation field.

2.1 Self-Organization

We must remind ourselves that what we are looking at is not the ground state of the system, but the stationary state. The condensate is coupled to a decaying cavity field and hence it loses energy. The system is dissipative owing to photon loss from the cavity. Thus we have to consider the steady state of a driven open system far from equilibrium and consider perturbations about this steady state. To begin with look at equations 1a, 1b reveals that they are invariant under the scaling transformation, $\alpha \to \alpha/\sqrt{N}$ as long as the parameters $NU_0, \sqrt(N)\eta, Ng_c$ are kept constant. Thus, we can incorporate the atomic number in the field amplitude and the system parameters aslong as they are expressed in the combination above. The realization of this fact helps in the numerical solution of these non linear coupled differential equations. In the steady state the system is characterized by the field amplitude α_0 and and condensate wave function $\psi(x,t) = \psi_0(x) \exp -i\mu t$. Substituting them in the above formula, and setting the $\frac{\partial}{\partial t}\alpha = 0$, we get the set of equations

$$\alpha_0 = \frac{N\langle \eta_t(x) \rangle}{\Delta_C - N\langle U \rangle + i\kappa} \tag{2a}$$

$$\mu\psi_0(x) = \left(\frac{p^2}{2\hbar m} + |\alpha_0|^2 U(x) + 2Re\{\alpha_0\}\eta_t(x) + Ng_c|\psi_0(x)|^2 \right) \psi_0(x) \tag{2b}$$

Here the expectation values have been calculated with respect to the steady state $\psi_0(x)$. These equations have been solved numerically. It is done by assuming some trial solution of the instantaneous $\psi_0(x)$ and then evolving it in imaginary time using equation 1b while at the same time adiabatically removing the cavity field using equation 2a. The advantage of using the imaginary time method is that all the energies higher than the ψ_0 i.e μ decay faster and are thus removed, the solution then converges to the steady state BEC wave function. At each step we make sure that the wave function is normalized.

On adiabatically eliminating the cavity field as described above the resultant potential in the GPE 1b is

$$V(x) = U_1 \cos(kx) + U_2 \cos^2(kx) \tag{3a}$$
$$U_1 = 2\langle\cos(kx)\rangle N I_0[\Delta_C - NU_0\langle\cos^2(kx)\rangle] \tag{3b}$$
$$U_2 = \langle\cos(kx)\rangle^2 N^2 I_0 U_0 \tag{3c}$$

It would serve to note that the potentials are defined in terms of the wave functions itself as far as the mean values are concerned. Hence, the equation is nonlinear. A trivial solution of the equation is the uniform state $\psi_0(x)$ =constant and $\alpha_0 = 0$,the potentials $U_1, U_2 = 0$ as $\langle\cos(kx)\rangle = 0$. However this does not imply that this is a stable solution, because fluctuations can drive the system away from this state as we will see for appropriate parameters.
Let us look at the behaviour of two quantities now that show the the self organization and localization of the wavefunction respectively.

$$\Theta = \langle\psi_0|\cos(kx)|\psi_0\rangle \tag{4}$$

. This is a good quantity to look at to understand self organization, for a uniform system $\Theta = 0$. For self-organization in which the atoms all congregate at the even sites $\theta = 1(kx = 2n\pi)$, when they are at the odd sites $\Theta = -1(kx = (2n+1)\pi)$.
Similarly, another quantity that describes how localized the atoms are can be described by the bunching parameter,

$$B = \langle\psi_0|\cos^2(kx)|\psi_0\rangle \tag{5}$$

Here, I_0 is the depth of the potential,it is given by

$$I_0 = \frac{|\eta|^2}{[\Delta_C - NU_0B]^2 + \kappa^2} \tag{6}$$

Self organization happens because of the U_1 term which is linear in $\cos(kx)$, the term $\cos^2(kx)$ does not discriminate between the odd and even sites and thus plays no role in the onset of self organization.
If we set $\Delta_C < -N|U_0|$(remember that $U_0 < 0$), the sign of U_1 is opposite to the sign of the Θ(refer to equation 3b). Thus, let us suppose that the fluctuations are such that there are more atoms in the even sites when compared with the odd site. Hence the sign of $\Theta > 0$. Hence, $U_1 < 0$. As a result, now the even sites are the minima of the U_1 potential, whereas the odd sites are the maxima. Thus, more of the atoms are attracted to the even sites, resulting in a runaway effect. On the other hand when fluctuations have more atoms at the odd sites when compared with the even sites,$\Theta < 0, U_1 > 0$, thus the odd sites are now the minima of the potential, and this encourages more atoms to go to the odd sites. The λ periodic lattice of condensate atoms fulfils the Bragg criterion of constructive interference and the atoms scatter pump photons into the cavity($\eta \neq 0$). There are of course dissipative processes(cavity loss) which take away the potential energy that gets converted into kinetic

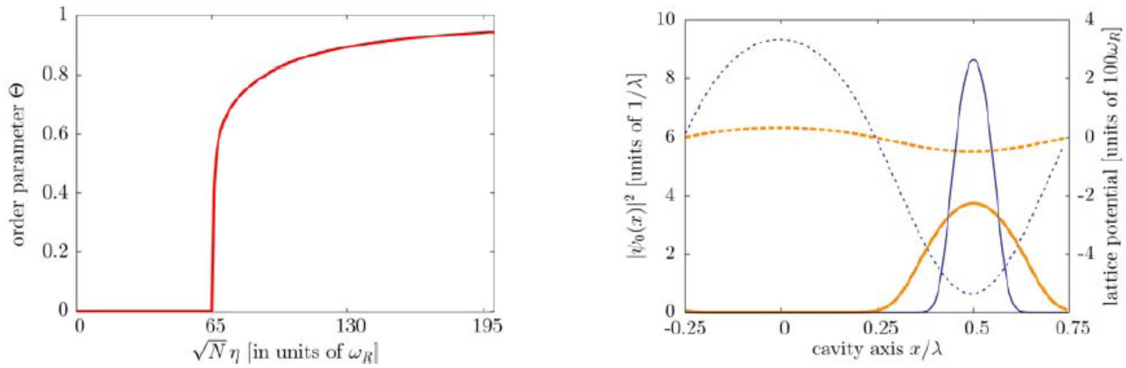

Fig. 1. (Color online) The order parameter Θ, plotted as a function of the transverse pump amplitude $\sqrt{N}\eta$, exhibits the self-organization of the BEC into a λ-periodic optical lattice. Parameters: $Ng_c = 10\hbar\omega_R\lambda$, $NU_0 = -100\omega_R$, $\Delta_C = -300\omega_R$, $\kappa = 200\omega_R$.

Fig. 2. (Color online) Two typical atomic position distributions $|\psi_0(x)|^2$ (solid lines) and the corresponding adiabatic optical potentials from equation (3) (dashed lines) for pump strengths $\sqrt{N}\eta = 100\omega_R$ (thick lines) and $\sqrt{N}\eta = 300\omega_R$ (thin lines). Other parameters are the same as in Figure 1.

Figure 2: taken from [7]

energy when the atoms fall into the wells. Also, the kinetic energy, interatomic interactions and collisions try to oppose the localization of the wave functions inside the cavity, and try to spread the wave function out.

2.2 Simulations and critical pump power

Now, let us look at the numerical solution of eqns 1a, 1b and see whether we get the self organization predicted above as we increase the pump power. The onset of self organization is indicated by the increase of the parameter Θ from 0 to 1. in the simulations the parameters are expressed in units of the recoil energy $\hbar\omega_r = \frac{\hbar^2 k^2}{2m}$ and the wavelength λ of the cavity field. Thus, when Θ is plotted as a function of the pump power $\sqrt{N}\eta$, above a critical value we see the onset of self organization as Θ increases from 0 to 1 as we see in Figure 2. This is accompanied by a spontaneous symmetry breaking as the atoms either occupy $kx = 0$or $kx = \pi$. The localization of the atoms is indicated when we plot $|\psi_0(x)|^2$. As we increase the pump power the atoms become more and more localized.
This reminds us of the Mott-insulator transition. In that case when the atom interactions are increased, multiple occupation of the sites is disfavoured , hence the system is driven to the localized phase with the atoms evenly distributed among the sites. However, when the kinetic energy dominates the interatomic interaction, the atoms are completely delocalized in the system leading to the superfluid state. In contrast for self organization the competition is between the kinetic energy and the energy of the cavity field. From the simulations, we see that at lower pump power the kinetic energy dominates, the condensate is completely delocalized. As the pump power is increased the atomic potential gradually deepens and the system becomes more and more localized. This is very nicely demonstrated in the results of the numerical simulations. The interatomic interaction also opposes the localization of the

wavefunction and tends to spread it out.
The critical pump power is determined by looking at perturbations to the uniform ground state and determining for what value of the pump power it becomes unstable.
Let us look at the equations 2a, 2b. The trivial state is given by, ψ_0=constant,$\alpha_0 = 0, \mu = Ng_c$. If we want to get a non zero value of the cavity field α_0 we must have a non zero perturbation of the wave function modulated by $\cos(kx)$ in the wave function ψ_0 as we see from equation 2a.

Hence the strategy is to introduce a small perturbation to the trivial wave function making it $\psi_0 = 1+\epsilon\cos(kx)$, evolve it in imaginary time and thus determine what range of parameters makes the perturbation grow. Thus we can understand at what set of parameters the trivial wave function is unstable to fluctuations of the length scale of the cavity wavelength. To do that we put the perturbed wave function in equation 2b and replace the photon no. α_0 in 2b from 2a. We get the equation

$$\frac{\Delta\psi}{\Delta\tau} = -Ng_c - \epsilon\cos(kx) \times \left\{\omega_R + N\eta^2 \frac{2\Delta_C - NU_0}{(\Delta_C - NU_0/2)^2 + \kappa^2} + 3Ng_c\right\} \tag{7}$$

From the above equation we see that as expected the homogenous part of the wave function decays at the rate Ng_c, whereas the perturbation decays at the rate given by the term in the brackets. In order that the fluctuation will lead to self organization, we require that the rate of decay of the fluctuation should be less than or in the critical case be equal to that of the homogenous condensate Ng_c. So we set the term in the brackets equal to Ng_c and get for the critical pump power

$$\sqrt{N}\eta_c = \sqrt{\frac{(\Delta_C - NU_0/2)^2 + \kappa^2}{2\Delta_C - NU_0}}\sqrt{\omega_R + 2Ng_c} \tag{8}$$

2.3 Connection with the Dicke Phase transition

The description in this section is drawn primarily from [1]. Let us talk about the relation of self organization to the Dicke phase transition. To draw a connection with the experimental evidence described in the next section, let us go back to the setup described in section 1.1 and in figure 1 where the cavity mode is along the x axis and the pump laser is along the z axis. Scattering between the pump field and the cavity mode creates a lattice potential in the x-z plane. The relative phase between the pump and the cavity field can either be 0 or π. When spontaneous symmetry breaking due to self organization happens the relative phase has value either 0 or π at all occupied positions. Now, analogous to the previous section, the order parameter describing the phase transition $\Theta = \langle\cos(kx)\cos(kz)\rangle$. $\Theta = \pm 1$(depending upon whether even or odd sites are preferentially occupied). In the dicke transition we have a system of two level atoms which are coupled to an electromagnetic field. for sufficient coupling the system enters a superradiant phase with macroscopic occupation of the field mode. Here we will not describe the Dicke model. Rather the path we take is to show that the model of self organization we have already shown is equivalent to the Dicke

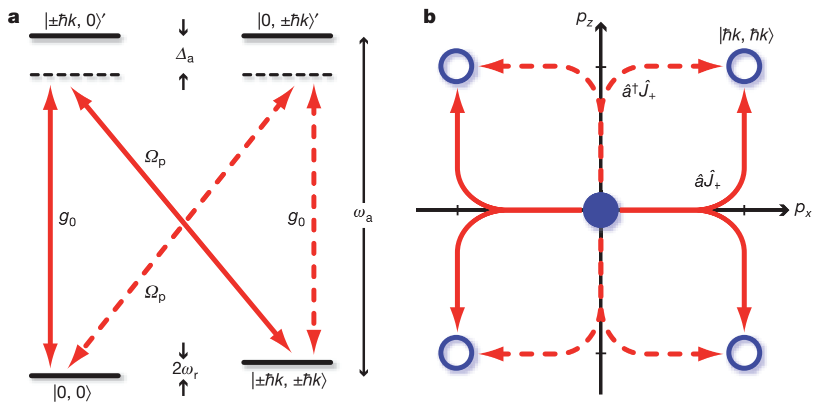

Figure 3: taken from [1]. Light scattering between the pump field and the cavity mode couples the zero momentum state $|0,0\rangle$ to the superposition of states $|\pm\hbar k, \pm\hbar k\rangle$. This can happen through two channels as shown schematically in the above diagram

model, and show how to recast the description in terms of a two -level system which above a critical coupling(critical pump power) goes into superradiant phase.

Intially the BEC is in the state $|p_x, p_z\rangle = |0,0\rangle$; due to photon scattering this state gets coupled to the symmetric superposition of states which carry equal momentum along the x and z directions. $\sum_{\mu,\nu=\pm1} |\mu\hbar k, \nu\hbar k\rangle/2$. The energy of this state is thus twice the recoil energy $E_r = \frac{\hbar^2 k^2}{2m}$ when compared with the zero energy state. It does so by two processe as shown is 3, a)absorption of a pump photon followed by emission into the cavity($a^\dagger J_+$) b) absorption of a cavity photon followed by emission into the pump, aJ_+. Here, $J_+ = J_-^\dagger = \sum_i |\pm\hbar k, \pm\hbar k\rangle_i \langle 0,0|$ So, the second quantized Hamiltonian has the term $(a^\dagger + a)(J^+ + J^-)$. which is the same as the Dicke Hamiltonian. At self organization both the cavity field and so to speak the 'atomic polarization',(as the higher energy state is occupied) acquire macroscopic values.

3 Experiments

In this section we describe the experimental evidence that has shown the existence of the self organized phase and corresponding supperradiant Dicke transition and shows the onset of one means the onset of the other. Although a number of experiments have been done to show self organization, **here we focus on the experiments reported in [1].**

These experiments are done with BEC's made with typically 10^5 Rb^{87} atoms trapped in a dipolar atomic trap which is centred inside a ultra-high finesse Fabry Perot cavity. The atoms are driven perpendicularly in the z direction by a red-detuned laser beam, the atom pump detuning is more than five orders of magnitude when compared with the atomic linewidth, and hence spontaneous emission can be safely neglected.

The light leaking out of the cavity is measured with a single photon counting module. The atomic momentum distribution is inferred from absorption imaging along the y axis after a few milliseconds of free ballistic expansion of the atomic cloud. For our purposes, this is as far as we will go into the details of the experimental setup.

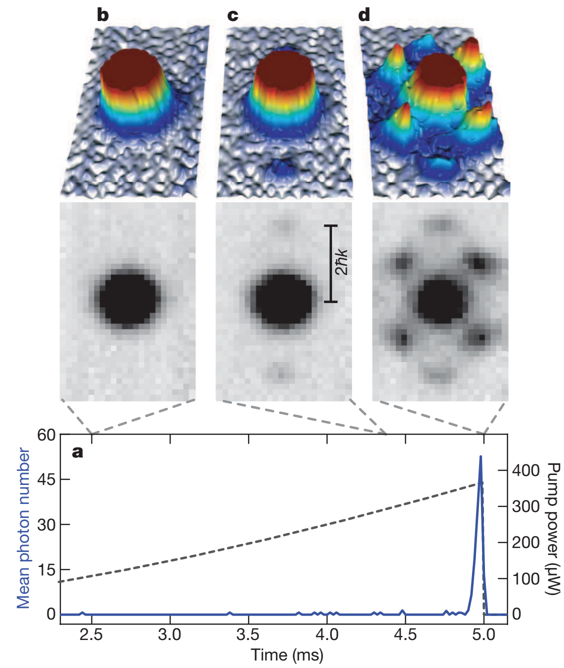

Figure 3 | Observation of the phase transition. a–d, The pump power (dashed) is gradually increased while the mean intracavity photon number (solid; 20-μs bins) is monitored. After the sudden release of the atomic cloud and its subsequent ballistic expansion for 6 ms, absorption images (clipped equally in atomic density) are made for pump powers corresponding to lattice depths of $2.6E_r$ (**b**), $7.0E_r$ (**c**) and $8.8E_r$ (**d**). Self-organization is manifested by an abrupt build-up of the cavity field accompanied by the formation of momentum components at $(p_x, p_z) = (\pm\hbar k, \pm\hbar k)$ (**d**). The weak momentum components at $(0, \pm 2\hbar k)$ (**c**) result from loading the atoms into the one-dimensional standing-wave potential of the pump laser. The pump–cavity detuning was $\Delta_c = -2\pi \times 14.9(2)$ MHz and the atom number was $N = 1.5(3) \times 10^5$ (parentheses show uncertainty in last digit).

Figure 4: Figure from [1]

Looking at figure 4 we see that at pump power below the critical value, the mean number of photons is negligible and the BEC has the expected uniform momentum distribution of the condensate loaded in a shallow optical cavity. However, as the pump is ramped up beyond the critical value we see a sharp increase in the mean number of photons in the system indicating a macroscopic occupation of the cavity field. Simultaneously, when we look at the momentum resolved images of the condensate we see that we have a sharp change with additional components appearing at $(\pm\hbar k, \pm\hbar k)$, exactly as we expected. Thus we have a density modulation as in a crystalline phase.

It was also found in this experiments that the self organized phase was stabilized by light scattering forces. However, if the pump intensity is kept constant we see a steady decrease in photon number, this was attributed to the atom loss caused by residual spontaneous scattering.

4 Conclusion

To conclude it has been shown how a BEC loaded in an optical cavity driven by a transverse laser field can undergo a spontaneous phase transition to either one of two symmetry broken states where either odd or even sites are occupied depending upon the fluctuations in the system, for a certain range of parameters and as long as the pump power is greater than a critical value.
It has also been shown that the intensity of the scattered light at the self organized state goes quadratically as the number of atoms. Following that lead, we have motivated the mapping of this system to the Dicke quantum phase transition.
At self organization the BEC enters a supersolid state with existence of non trivial diagonal long range order associated with density modulation and off diagonal long range order associated with phase coherence. Along with that an experimental finding was reported where this self organization was observed in the momentum resolved imaging after ballistic expansion.
However, it should be noted that the whole analysis was done at the mean field level, to obtain a more complete picture we must understand whether the self organized state is stable to fluctuations and interactions beyond the mean field treatment.
The other important consideration is determining the quantum depletion and whether phase coherence is there as the atoms are kicked out of their condensed state due to atom atom and atom light interactions. Then, the validity of the method used would be questionable. An analysis of depletion with the cavity linewidth set to zero was performed in [7], it was found that the number of noncondensed atoms indeed blows up at the critical power. This indicates that the present analysis is ok except in a small region around the critical point. A more systematic analysis with the cavity mode linewidth $\kappa \neq 0$ needs to be performed.

Finally, it should be noted that other work has been done in [3] where the phenomenon of BEC trapped in a cavity coupled to a continuum of modes is considered. In this case the transition to the ordered phase is associated with the breaking of a continuous translational symmetry as in conventional crystalline ordering. This opens up avenues for realizing other phenomenon like defects, grain boundaries and dislocations.

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