Quantum Phases and Topological States in Optical Lattices

Pak On Chan
Department of Physics, University of Illinois at Urbana-Champaign
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Abstract

Optical lattices are periodic potentials giving rise from optical fields. In this review essay, system of cold bosonic atoms sitting in an optical lattice is considered. Different quantum phases in optical lattices are discussed. In particular, we will talk about the superfluid and the Mott insulating phases of the system. Finally, a brief introduction on the topological states found in the optical lattices is given.

1. Introduction to Optical Lattices

In the construction of an optical lattice, neutral bosonic atoms are placed in an external electric field $\vec{E}$ provided by optical beams with frequency ω. Though the atoms are charge neutral, they interact with the electric field $\vec{E}$. As the optical field is turned on, the neutral atoms are polarized under the field. The induced dipoles on the atoms in turn provide a bridge for the atoms to interact with the electric field $\vec{E}$. The energy V associated with a dipole interaction is given by

$$V = -\vec{d} \cdot \vec{E} = \alpha(\omega) E^2 \tag{1}$$

where $\vec{d}$ is the induced dipole on the atom and the equal sign hold since $d = \alpha(\omega)\vec{E}$ where $\alpha(\omega)$ is the polarizability of the atom. In the large detuning limit, there is no absorption and emission, so $\alpha(\omega)$ can be considered as real. Since the electric field field $\vec{E}$ is a function of position $\vec{r}$, the energy V lead to a potential for the neutral atoms. Such phenomenon is in general known as the stark effect [1]. Note that the intensity I is proportional to E^2, so the strength of the potential can be changed by altering the intensity of the light beams.

A periodic potential V can be generated by a standing wave of electric field , practically constructed by two counter propagating beams of the same amplitude and wave number k. Two counter propagating beams in one direction creates single standing wave (see Fig.1a), giving rise to wells in the form of array of planes. Standing waves in two perpendicular directions (see Fig.1b) result in wells like array of lines. Note that the interference between the beams can be eliminated by suitably choosing the orthogonal plane polarizations. Similarly, three orthogonal standing waves (see Fig.1c) generate trapping wells in the form of simple cubic lattice of points [1]. In such case, the electric potential V_{ext} for a single particle takes the simple form as

$$V_{ext}(\vec{r}) = V_o \sum_i \sin^2 kx_i \tag{2}$$

where V_o is the depth of the potential. By controlling the beam intensity I, the well depth V_o can be tuned experimentally. Note that the lattice constant a of such simple

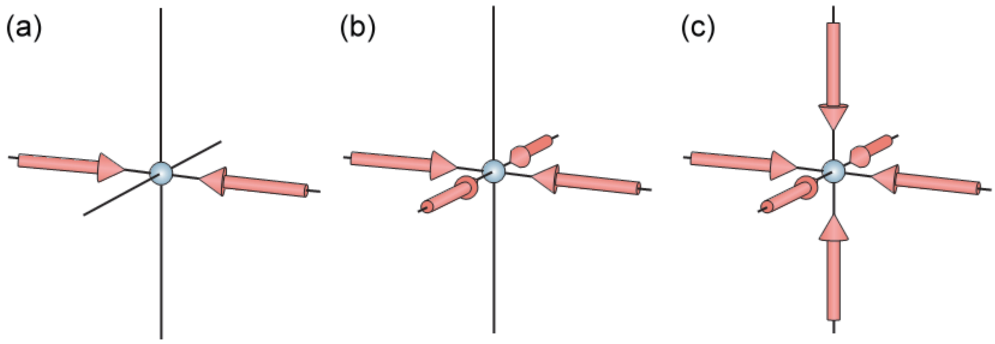

Figure 1: A neutral atom is sitting in an optical field. (a) Standing wave in one direction gives rise to potential like array of planes. (b) Standing waves in two perpendicular directions result in potential like array of lines. (c) Standing waves in three orthogonal directions generate potential like simple cubic lattice.

cubic lattice is given by $a = \pi/k$, so the lattice spacing can be changed by controlling the wavelength of the beams. Indeed, one can also achieve different lattice configurations by changing the angles between the standing waves.

Practically, the laser beams used in the experiments are not perfect perfect plane waves. Instead, they are gaussian beams. The effect of gaussian envelope consequently leads to an additional slowly varying trapping potential. Since such effect is not related to the main idea of the topic, we will omit the effect in the discussion.

Since all the parameters in the lattice structure can be tuned experimentally by controlling the optical beams, optical lattice provide us with a controlled way to construct lattice structures typically found in condensed matter system. Furthermore, our lattice structure is longer restricted to the materials found in the laboratories. By controlling the parameters and the structure of the optical lattice, different emergent state of matter can be manifested. In the following, we will talk about how we can construct different quantum phases in the optical lattices. Also, we will discuss the topological states recently found in the optical lattices.

2. Quantum Phases

Quantum phase is the state of matter at absolute zero temperature in which the thermal energy is vanishing. Without thermal energy, there is no longer thermal fluctuation, explicitly, all the Boltzmann factors in the density matrix die out except the factor for the ground state. Hence, state of matter at absolute zero temperature refers to the ground state of the system. Recall that classical phase transition is the change between a disordered phase and an ordered phase due to the change of temperature. In contrast, quantum phase transition is the change of phase at the fixed absolute zero temperature. The variable that drives the transition is the parameter in the hamiltonian, for example, the external magnetic field or applied pressure on the a system.

In the following, we will discuss the quantum phases in a system of cold atoms staying in an optical lattice. To be more specific, we will talk about the superfluid phase and the

Mott insulating phase in the system. The first step of the discussion is to introduce the Bose-Hubbard model, which quantum mechanically describes a system of cold atoms in an optical lattice.

2.1 Bose-Hubbard Model

In Section 1. we have discussed the potential experienced V_{ext} by a neutral atom in the optical lattice. Suppose we put N atoms into the optical lattice, in addition to the periodic potential, the atoms will experience scattering due to collision with neighboring particles. Though neutral atoms can interact through the van der Waals interaction, the dominating scattering process is the s wave scattering between atoms since s wave scattering corresponds to a much longer scattering length a_s. In such case, van der Waals interaction plays only a minor role in the two body interaction. Hence the interaction V_{int} between an atom at $\vec{x}_1$ and an atom at $\vec{x}_2$ can be effectively written as

$$V_{int}(\vec{x_1} - \vec{x_2}) = g\delta^3(\vec{x_1} - \vec{x_2}) \tag{3}$$

where the coupling strength $g = 4\pi\hbar^2 a_s^2/m$ and m is the mass of the atoms [2]. With the external electric potential V_{ext} and the s wave scattering potential V_{int}, one can describe the system quantum mechanically. Such desciption of the system is known as the Bose-Hubbard model. In the language of second quantization, the many body hamiltonian of atoms in an optical lattice is given by the typical form

$$\begin{aligned} \hat{H} &= \int \hat{\psi}^\dagger(\vec{x})\Big(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{x})\Big)\hat{\psi}(\vec{x})d^3x \\ &+\frac{1}{2}\int\int \hat{\psi}^\dagger(\vec{x}_1)\hat{\psi}^\dagger(\vec{x}_2)V_{int}(\vec{x}_1 - \vec{x}_2)\hat{\psi}(\vec{x}_2)\hat{\psi}(\vec{x}_1)d^3x_1 d^3x_2 \end{aligned} \tag{4}$$

where $\hat{\psi}(\vec{x})$ and $\hat{\psi}^\dagger(\vec{x})$ are the field operator. $\hat{\psi}(\vec{x})$ annihilates an atom at $\vec{x}$ whereas $\hat{\psi}^\dagger(\vec{x})$ creates an atom at $\vec{x}$. Since the atoms we are considering are bosons, the field operators obey the (equal time) commutation relation $[\hat{\psi}(\vec{x}_1), \hat{\psi}^\dagger(\vec{x}_2)] = \delta^3(\vec{x}_1 - \vec{x}_2)$. Assume that the atoms are cold enough such that higher bands are irrelevant to the physics involved, then we can simply consider the lowest band in the system [2].

To make the physical picture more transparent, we introduce the set of Wannier functions $\{w(\vec{x} - \vec{x_i})\}$ as the orthonormal basis for the lowest band [2, 3]. Note that the Wannier function $w(\vec{x} - \vec{x_i})$ is a superposition of Bloch waves and is localized at the well labeled by i. Since the Wannier functions is complete, we have the fourier expansion for the field operator

$$\hat{\psi}(\vec{x}) = \sum_i \hat{a}_i w(\vec{x} - \vec{x_i}) \tag{5}$$

where $\hat{a}_i$ annihilates a particle in the mode i with mode function $w(\vec{x} - \vec{x_i})$, and the hermitian conjugate $\hat{a}_i^\dagger$ does the reverse process. They obey the commutation relation $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. By substituting the expression for $\hat{\psi}(x)$ into Eq.(4), the Bose-Hubbard hamiltonian becomes

$$\hat{H} = -J\sum_{<ij>} a_i^\dagger a_j + \frac{1}{2}U\sum_i \hat{n}_i(\hat{n}_i - 1) \tag{6}$$

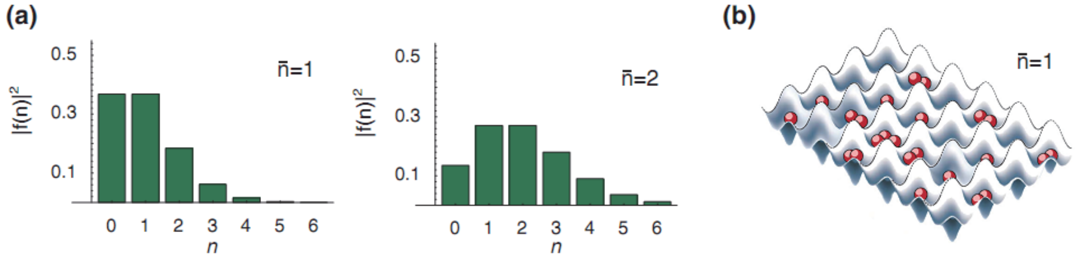

Figure 2: Poissonian distributions for the number of particle in a site is shown in the superfluid ground state [5]. (a) Probability distribution for $\langle n \rangle = 1$ and $\langle n \rangle = 2$ for a certain cite. (c) Heuristic picture of the superfluid ground state in the optical lattice for $\langle n \rangle = 1$.

where $n_i = a_i^\dagger a_i$, tunneling matric element $J = -\int w(\vec{x} - \vec{x_i})(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{x}))w(\vec{x} - \vec{x_j})d^3x$ and the interaction energy $U = g\int |w(\vec{x})|^4 d^3x$ [2, 3, 4]. As seen in the expression, the Hamiltonian consists of two crucial terms. The first term in the hamiltonian is known as the hopping term which describes the tunneling of atoms between neighboring sites. The strength is characterized by the tunneling matric element J which is generally positive. A single particle state with non-zero components on neighboring sites gives a more negative hopping energy. Hence, the hopping term tends to delocalized the single particle wave function in the optical lattice [5]. The second term in the hamiltonian is known as the interaction term which describes the on-site interactions of atoms. Since U specifies the interaction energy between two atoms staying in the same site. The component $U\hat{n}_i(\hat{n}_i - 1)/2$ counts the total interaction energy in the $i - th$ site and the sum counts all the on-site interaction energys in the lattice. A many particles state with atoms well separated in different sites gives the minimal interaction energy which is zero. Hence the second term in the hamiltonian tends to localize atoms in different sites [5].

Note that once the periodic potential is fixed, the value of J and U is determined. If we increase the potential depth V_o by tune up the intensity of the optical beams in the experiment, it is less likely for atoms to tunnel through the barrier. Hence, it is expected that the tunneling matrix element J decreases as the potential depth V_o increases. On the other hand, the atoms are tightly squeezed together. Thus, it is expected that interaction energy U between two atom in a well increases as the well depth V_o increases. Notice that the only parameter which determines the ground state of the Bose-Hubbard hamiltonian is the dimensionless ratio U/J. We will see that the ground states of the system have completely different properties in the two extreme regime (i) $U/J \ll 1$ and (ii) $U/J \gg 1$.

2.2 The Superfluid Phase

Consider the case in which the tunneling matrix element J is much larger than the interaction energy U, i.e. $U/J \ll 1$. To get a sense of the ground state in such case, it is better to start with the case where $U = 0$. The problem then reduces to a system

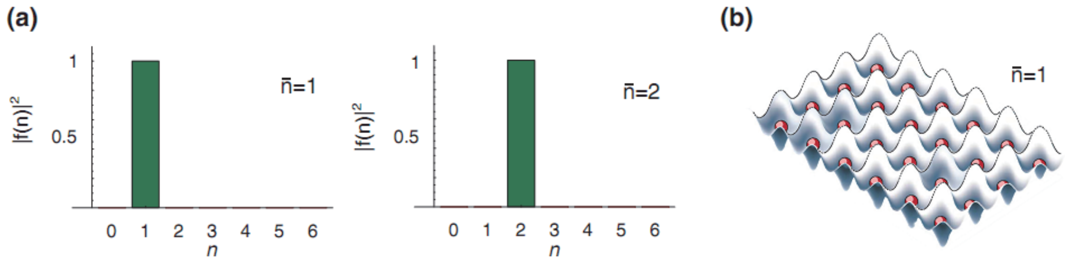

Figure 3: Poissonian distributions for the number of particle in a site is shown in the Mott insulating state [5]. (a) Probability distribution for $\langle n \rangle = 1$ and $\langle n \rangle = 2$ for a certain site. (c) Picture of the Mott insulating ground state in the optical lattice for $\langle n \rangle = 1$.

of N non-interacting bosons in a periodic potential. At zero temperature, Bose-Einstein condensation happens, consequently, each atom is condensed to the ground state of the corresponding single particle hamiltonian. Thus, the total ground state is given by the product state of N particles each of them are staying in the Bloch state of zero crystal momentum in the lowest band. Hence each atoms are completely delocalized in the optical lattice, meaning that the wave function of single particle spread over the whole lattice. It is realized as a superfluid (SF) phase. In terms of the field operators a_i's introduced previously, the ground state is

$$|\Psi_{SF}\rangle = \mathcal{N}\Big(\sum_i a_i^\dagger\Big)^N |0\rangle \approx \mathcal{N}' \prod_i |\alpha_i\rangle \tag{7}$$

where $\mathcal{N}$, $\mathcal{N}'$ are some normalization constant and $|0\rangle$ is the vacuum state [3, 5]. The state $|\alpha_i\rangle$, where $\alpha_i \in \mathbb{C}$, on the right hand side stand for a coherent state. Effectively, there is a coherent state associated with each lattice site. Consequently, the probability distribution of $\hat{n}_i$ is a Poissonian with variance $Var(\hat{n}_i) = \langle n_i \rangle$ which is non zero in each site. Fig.2a shows the probability distributions for $\langle n_i \rangle = 1$ and $\langle n_i \rangle = 2$ respectively. Fig.2b heuristically illustrate the delocalized atoms in the superfluid phase.

For the case of $U/J \ll 1$ with non zero U, the problem can be solved by introducing the Bogoliubov transformation and the ground state will then be the state with no quasi particle. However, such ground state breaks the $U(1)$ gauge symmetry of the hamiltonian. Hence, it is expected that there is one massless goldstone boson associated with such broken symmetry and we can have gapless excitations. Besides, macroscopic wave function $\psi(\vec{x}) = \langle \hat{\psi}(\vec{x}) \rangle$ can be defined in such weakly interacting regime [6]. The problem will then become solving the Gross-Pitaevskii equation for the mean field $\psi(\vec{x})$.

2.3 The Mott Insulating Phase

Now, we consider the limit in which the interaction energy is much larger than the tunneling matrix element J, i.e. $U/J \gg 1$. Similarly, it would be easier to start with the case where we have infinite well depth V_o in which $J = 0$, meaning that the atoms can never tunnel through the barrier. The only term left in the hamiltonian is the interaction term and it is already diagonalized. To minimize the on-site interaction, we should never put two particles on the same site unless it is really inevitable. More generally, we should put equal number of particles in each site so as to minimize the total energy. Such ground state is realized as the Mott insulating phase. The ground state of the system is then given by

$$|\Psi_{SF}\rangle = \mathcal{N} \prod_i \left(a_i^\dagger\right)^n |0\rangle \tag{8}$$

where $\mathcal{N}$ is some normalization constant and n is the mean number of particle per site [3, 5]. Note that such ground state is a product state of Fock states of the sites, so each site has well defined number of particle with no fluctuation. Fluctuation in number of particle in a site becomes s very energetically costly in this case. Fig.3a shows the probability distribution for $\langle n_i \rangle = 1$ and $\langle n_i \rangle = 2$ respectively. Also Fig.3b pictorially shows the localized atoms in the Mott insulating phase.

Since the ground state has well defined number of particles in each site, the expectation value $\langle \hat{a}_i \rangle = 0$ and hence $\langle \hat{\psi}(\vec{x}) \rangle = 0$. Hence we cannot establish a coherent macroscopic wave function in the Mott insulating phase. In other words, fixing the number of particles in the sites lose the coherence. So the many body problem cannot be solved by the theory of weakly interacting bosons established by Bogoliubo, Gross and Pitaevskii [5].

Unlike the case in superfluid regime where there is gapless excitation, the lowest lying excitation in the Mott insulating regime is gapped. In the limit of $U/J \gg 1$, the energy gap $\Delta = U$ [5]. Such phenomenon can be understood by considering a Mott insulating ground state in which each site is filled with one atom. The lowest lying excitation corresponds to removing an atom from a site, and then place it in another site. The energy cost in taking away an atom is zero, but the amount of energy that we need to impart into the system so as to place the taken atom into another site, which is already occupied by another atom, is U. That explains why the energy gap for the lowest lying excitation equals U. Such process is highly analogous to particle-hole pair creation.

2.4 Quantum Phase Transition

We have just seen that the state of matter at zero temperature in the two extreme regime (i) $U/J \ll 1$ and (ii) $U/J \gg 1$. The former case gives a superfluid phase whereas the later case gives a Mott insulating phase. In between the two regime, there is a critical point at which quantum phase transition occurs. If the mean number $\langle n \rangle$ of particles in each site is much larger than unity, meaning that many atoms are embedded into the optical lattice, then the quantum critical point is given by $(U/J)_c = 4z\langle n \rangle$ where $z = 2d$ is the number of neighboring sites for each site [5]. As we tune up the intensity of the light beams, the parameter (U/J) increases accordingly. At the quantum critical point $(U/J)_c$, quantum phase transition occurs. The atoms in the optical lattice will transit from the superfluid phase to the Mott insulating phase.

Experimentally, there are many ways in visualizing the quantum phase transition. All of them make use of the dramatic difference in properties in the two phases. Here we will discuss the experiment in observing the multiple matter wave interference pattern

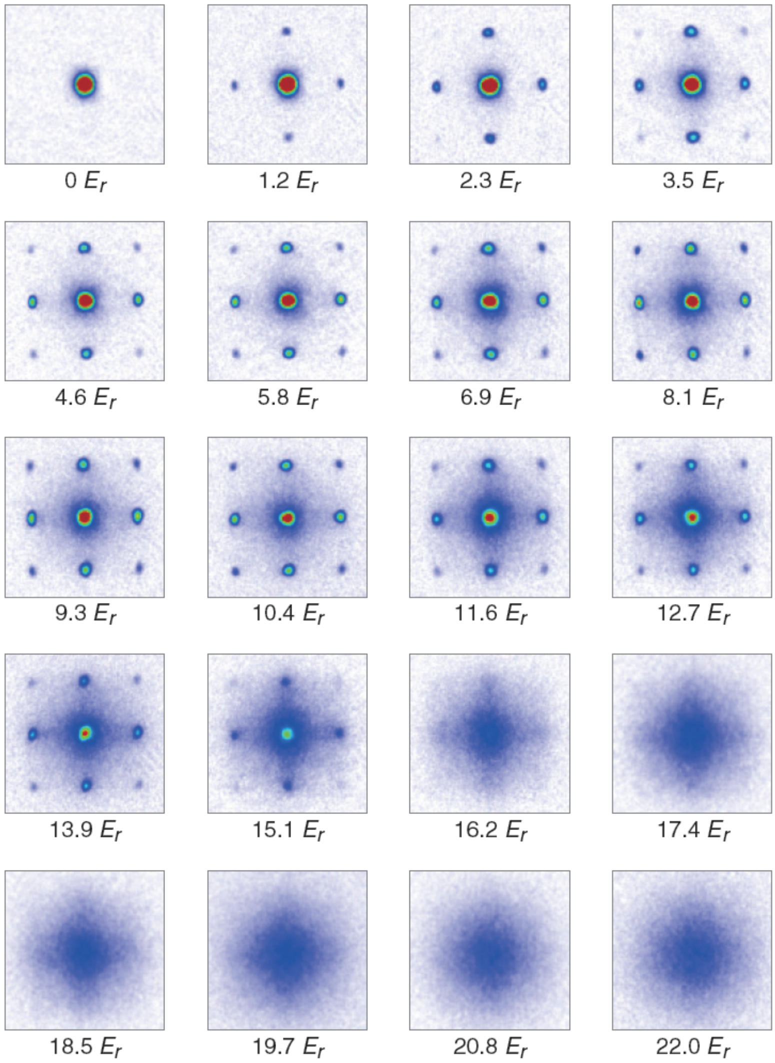

Figure 4: Multiple matter wave interference pattern observed at $t = 15ms$ [5]. Sharp interference is observed for $V_o < 10E_r$. The pattern becomes blurred if V_o is further increased. At about $V_o = 20E_r$, the interference pattern vanishes.

[3, 5]. About 10^5 number of ^{87}Rb atoms are immersed into the optical field with trapping frequency 240 Hz. The well depth V_o is measured in unit of recoil energy $E_r = \hbar^2 k^2/2m$. By controlling the well depth V_o, the bose gas is initially prepared in different phases. The optical potential is turned off suddenly, and the gas of atom is then allowed to evolve freely. As seen in Fig.4, sharp interference pattern is observed when $V_o < 10E_r$, which corresponds to the superfluid coherent phase. An incoherent background emerges when V_o is increased further. Finally, no interference pattern can be observed and it corresponds to the complete Mott insulating phase.

There are various methods in visualizing the quantum phase transition in optical lattices. For example, an experiment can be done in probing the critical velocity of the bose gas. Finite critical velocity will be found in the superfluid phase but no in the Mott insulating phase. An experiment can also be performed in probing the excitation spectrum of the gas of atoms. Finite energy gap will be found in the Mott insulating phase but not in the superfluid phase.

3. Topological States

In this section, we will talk about the topological states recently found in the optical lattices [7]. Topological states in a quantum system refer to states in the energy spectrum which can not be eliminated by adding local perturbations that preserves the symmetry of the system. For example, the chiral edge states in two dimensional topological insulators with particle-hole symmetry is a topological state. Their spatial wave function is localized at the boundary and hence they are called edged states. They are chiral in the sense that they propagate only in one direction along the boundary of the quantum hall material. They cannot be continuously deformed away from the spectrum by locally perturbing the system at the same time obeying the particle-hole symmetry.

In the following, we will discuss the chiral edge states in an optical lattice. Again, it is the easy-tunable structures of optical lattices enable the observation of these emergent states in the laboratory. Before discussing the topological states, we will introduce the Haldane Model which describes a topological insulating system in an optical lattice.

3.1 Haldane Model

It is worth noting that topological states exist even when is no interaction. So, for simplicity, we do not include the small interaction between atoms here. We will work with a two dimensional system in x-y plane. Here, in additional to the periodic optical lattice potential V_{ext}, we consider a light-induced periodic vector potential $\vec{A}$, with zero total flux, upon the system [8]. The aim of the introduction of such vector potential will clear later. Furthermore, we add an extra confining electric potential V_c which serve to creat a boundary in the system. Explicitly, $V_c = 0$ for $r = 0$ and $V_c = \infty$ for $r > R_o$ [7].

Since the atoms in the lattice are non-interacting, they cannot see each other in the lattice. To solve for the band structure, it is suffice to work with the single particle hamiltonian. With the potentials V_{ext}, $\vec{A}$ and V_c, the single particle hamiltonian $\hat{H}$ is

$$\hat{H} = \frac{1}{2m}\left(\vec{p} - \alpha\vec{A}(\vec{x})\right)^2 + V_{ext}(\vec{x}) + V_c(\vec{x}) \tag{9}$$

where α is the induced coupling strength with the gauge field $\vec{A}$ [7, 9]. With the single particle hamiltonian $\hat{H}$, the lowest band of the system can be obtained by using s wave tight binding approximation.

3.2 The chiral edge states

The lowest band structure associated with the single particle hamiltonian in Eq.(9) has been solved numerically [7, 9]. The density of states and the spectrum are shown in Fig.5. Fig.5a shows the density of states of the band with $\alpha = 0$. It can be seen that there is no gap in the band constructed by the s wave tight binding approximation without the light-induced vector potential $\vec{A}$. Fig.5b shows the density of states of the

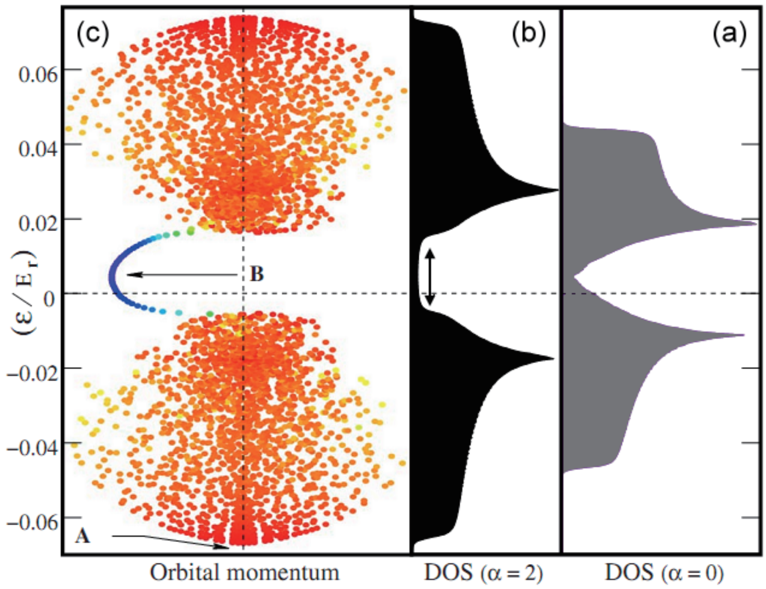

Figure 5: Density of states and the spectrum for the Haldance model [7]. Fig.5a shows the density of states of the gapless band with $\alpha = 0$. Fig.5b shows the density of states of the gapped band with $\alpha = 2$. Fig.5c shows the energy spectrum against the orbital angular momentum with $\alpha = 2$. A indicates the ground state and B indicates the chiral states

band with $\alpha = 2$. It can be observed from the figure that a energy gap is opened when light-induced vector potential $\vec{A}$ comes in. Only little amount of state is available in the gap.

Fig.5c shows the energy spectrum against the orbital angular momentum with $\alpha = 2$. It can be seen that there are states in between the energy gap. Note that these states are not symmetric about the vertical line with vanishing angular momentum. Such phenomenon is expected since the light-induced gauge field $\vec{A}$ breaks the time reversal symmetry of the system. Energy eigenstates may not comes in pairs with opposite angular momentum. These gapless states in the gapped are hence called chiral states. In the figure, A indicates the ground state of the system while B indicates the chiral states within the gap. Since the band structure resembles the form the topological insulator in which there is gapless edge states upon a gapped bulk band, such emergent phase in optical lattices is called the topological insulating phase.

With the single particle hamiltonian in Eq.(9), one can also solve for the energy eigenstates for the spectrum using the tight binding approximation. Let $\psi_n(\vec{x})$ be the eigenstate with energy ε_n. $|\psi_n(\vec{x})|^2$ gives the (probability) density distribution of the particle. Fig.6 shows the density distribution for the ground state A and the chiral state B respectively. Higher density is indicated by a brighter color. It can be seen that the ground state wave function is spread on the sites over the lattice. For the chiral states, the wave function is localized at the boundary of the lattice. Hence these states are the topological chiral edge state. It has been proposed that if atoms are loaded to the edge

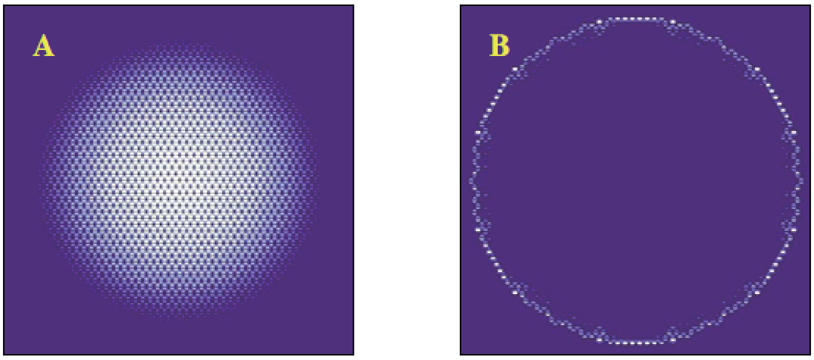

Figure 6: Density distribution $|\psi_n(\vec{x})|^2$ for the ground state A and the chiral state B respectively [7]. Higher density is indicated by a brighter color.

states, one can observe mass current along the boundary by imaging the atoms [7]. Since these states are chiral, the atoms in the edge states will only travel in one direction along the boundary.

It has been found that the boundary provided by the hard wall confining potential V_c plays an important role in the formation of the chiral edge states [7]. If one softens the boundary, then one cannot arrive at any edge states. The chiral edge states are found only if the thickness of the boundary is of length scale of several lattice constant a. If we soften the wall to a thickness of more than several lattice constant a, the edge states will mix with the Tamm states and eventually destroy the topological insulating phase in the optical lattice [7].

4. Conclusion

We have discussed the construction of optical lattice. We talked about the interacting bose gas of atoms in the optical lattice by introducing the Bose-Hubbard model. It is found that the system shows two quantum phases with completely different physical properties. They are the superfluid phase and the Mott insulating phase. We have also talked about experiments in visualizing the quantum phase transition at the critical point. Lastly, we discussed the topological chiral edge states theoretically found in the Haldane model. It is expected that there will be upcoming experiments verifying the existence of these topological edge states in optical lattices.

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