The Emergence of Criticality in the Earth's Atmosphere

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Abstract

The theory of self-organized criticality has been shown to have applications to many complex phenomena in nature. This paper will address the emergence of criticality in the Earth's atmosphere as observed in rainfall. Recent observations have shown power-law scaling of rainfall events, a signature of criticality. These results have been reproduced by means of a simple sandpile model which sheds light on the origin of criticality in the atmosphere. A summary of the model and a comparison of its results to observational data will be presented, followed by a discussion of findings and ideas for future research.

1 Introduction

Despite centuries of meteorological study, the Earth's atmosphere has proven to be a system of overwhelming complexity, rebuffing mankind's attempts to understand and accurately predict even the most common phenomena. Complications such as turbulence and sensitivity to minor perturbations make accurate prediction of the weather, on any timescale, very difficult. The atmosphere's chaotic nature means that small errors in model parameters can quickly transform into noticeable effects as time evolves. An accurate model with few input parameters would reduce the possibility for such errors. The study of emergent phenomena has provided a framework for studying complex systems with simple models, although the application of this framework to meteorology has been limited up to this point.

The atmosphere is a veritable playground of emergence: clouds, cold fronts, the jetstream, hurricanes, and the list goes on. Although the study of emergence has burgeoned within physics, little attention has been paid to emergence in the atmosphere. This paper will provide a brief summary of recent research on atmospheric precipitation, emphasizing the underlying collective phenomena involved without resorting to complicated many-parameter models. Much of the recent research in this area was prompted by a paper by Bak, Tang, and Wiesenfeld entitled "Self-Organized Criticality" [1], which will be explained briefly in the following section. It should be emphasized, however, that rather than focusing on the details of self-organized criticality or the properties of critical systems, this paper will focus primarily on collective processes which *lead* to criticality in rainfall.

The atmosphere makes an excellent testing ground for these ideas because of its availability for observation. Through observations, simple models can be tested and tweaked to match observational data. This paper will examine a very simple model for atmospheric dynamics and then compare model data with observational data.

1.1 Self-Organized Criticality

In 1988, a paper was published by Bak, Tang, and Wiesenfeld, which claimed to provide a framework for understanding many complex phenomena in nature. They called this phenomenon self-organized criticality (SOC), and stated that it "might be *the* underlying concept for temporal and spatial scaling in dissipative non-equilibrium systems" [1].

The idea behind SOC is that certain spatially extended, multidimensional, dynamical systems tend to organize themselves into a critical state, where critical is used in the same sense as in phase transitions. These systems are characterized by local interactions and some mechanism for energy addition (drive) and loss (dissipation). This is fundamentally different from a laboratory phase transition, when a parameter is actively tuned using some sort of experimental apparatus. In SOC systems, the parameters are tuned by the dynamics in such a way that the system remains in the vicinity of a critical point.

In the theory of phase transitions, critical points are marked by spatial self-similarity and power-law scaling of the order parameter. These are the signatures of SOC in non-equilibrium systems, and they provide evidence for SOC when found in nature, as in the case of rainfall.

1.2 The Sandpile Model

The sandpile model was proposed the in original SOC paper as a system which demonstrates the emergence of criticality through simple local interaction rules. In general, sandpile models start with a lattice of points, each of which contains a pile of sand. At each turn, an extra grain is added to one of the points. If the height difference between any one point and the point(s) next to it becomes greater than some predetermined critical value, then the sand at that point topples onto the surrounding points. Bak, Tang, and Wiesenfeld, presented a specific form of this model, and showed that the one-dimensional version did not organize itself into a critical state, but the 2- and 3-dimensional versions did exhibit SOC [1].

The 2-dimensional version of this model has been altered slightly to serve as a model for rainfall [2]. In this altered model, also referred to as the 2d Oslo model, instead of each lattice point being assigned a height, each point is simply assigned a height gradient. The gradient at lattice point (i, j) is denoted $z_{i,j}$. At the beginning of each turn, a grain of sand is added to every point on the $x = 0$ line:

$$z_{0,j} \rightarrow z_{0,j} + 1 \tag{1}$$

If the gradient value exceeds the predefined critical value z_c at any lattice point, the points around it are updated in the following way:

If $z_{i,j} \geq z_c$, then

$$\begin{aligned} z_{i,j} &\rightarrow z_{i,j} - 4 && (2)\\ z_{i\pm1,j} &\rightarrow z_{i\pm1,j} + 1 && (3)\\ z_{i,j\pm1} &\rightarrow z_{i,j\pm1} + 1 && (4) \end{aligned}$$

No more grains are added until the system is once again stable, that is, until $z_{i,j} < z_c$ for all i, j. Dissipation occurs through the boundaries at $x = 0$ and $x = x_{max}$. Periodic boundary conditions are imposed in the y-direction.

Though this model differs somewhat in appearance from the Bak-Tang-Wiesenfeld model, it retains the essential features of SOC, and it has the advantage of being more readily related to the physics of clouds and precipitation. Indeed, one virtue of sandpile models is their adaptability to different physical situations. In this paper, only its application to rainfall processes will be considered. Section 2.2 contains data produced by this model.

1.3 A Mean-Field Perspective of the Sandpile Model

It has been shown that the apparently parameter-less sandpile models presented above are in fact limiting cases of a more general mean-field model [3] [4]. Unsurprisingly, they represent the limiting case which leads to criticality. The following mean-field perspective of the models allows a closer look at their validity as realistic approximations. It was motivated by the fact that simple sandpile models violate locality by halting the addition of energy until after all avalanche events have ceased. In order to apply the tools of non-equilibrium physics, locality had to be restored by including a finite rate of energy addition.

The following is a summary of the derivation presented in [3]. In this model, instead of new grains being added systematically in the same place at each turn, grains are added to each lattice point with probability h at each time step. It should be noted that whereas in the models of section 1.2 a time step consisted of the duration of all avalanche events following an addition of energy, this mean-field model advances time at a constant rate, independent of activity in the system. Dissipation is simulated by removing a fraction ϵ of the grains originating at each toppling event. This is not totally different from the Oslo model, where energy was dissipated through the boundaries. The exact integer values $z_{i,j}$ for each lattice point are replaced by assigning each point to one of three categories: stable ($z_{i,j} < z_c - 1$), critical ($z_{i,j} = z_c - 1$), or active ($z_{i,j} \geq z_c$). The density of each type of point is labeled accordingly: ρ_s, ρ_c, and ρ_a. For each density, a rate equation can be written in the form:

$$\frac{\partial}{\partial t}\rho_i = \mathcal{F}_i(\rho_a, \rho_c, \rho_s) \qquad\qquad i = a, c, s \tag{5}$$

The time derivatives can be set to zero in order to find a stationary state, and two of the three equations of the form 5 can be combined with the condition that $\rho_a + \rho_c + \rho_s = 1$ to find solutions for each density. The function $\mathcal{F}_i$ can be expanded in the following way (where the constant term is removed in order to achieve stationarity):

$$\mathcal{F}_i = \sum_n f_i^n \rho_n + \sum_{n,l} f_i^{n,l} \rho_n \rho_l + \mathcal{O}(\rho_n^3) \tag{6}$$

Because each density is less than one, all terms of cubic order and higher can be dropped. The two sums in the above equation have different interpretations. The first represents the extent to which the density ρ_i is converted to another type of density in a single time step due to the rules of the model. The second sum represents the extent to which densities are changed due to interactions between lattice points. These terms come from toppling events in which nearest neighbor points are affected.

The function $\mathcal{F}_a$ will now be explained. The terms from the first sum are as follows:

$$\sum_n f_a^n \rho_n = -\rho_a + h\rho_c \tag{7}$$

These terms come from the removal of all active sites due to toppling and the conversion of critical sites to active sites due to the random addition of grains at each time step.

The terms from the interaction sum are:

$$\sum_{n,l} f_a^{n,l} \rho_n \rho_l = (g-\epsilon)\rho_c\rho_a + \mathcal{O}(\rho_a^2) \tag{8}$$

where g is a parameter dependent on the exact specifications of the model. Terms of order ρ_a^2 will be ignored in this analysis since at the critical point, $\rho_a \to 0$.

A similar analysis can be done to find $\mathcal{F}_c$, yielding two equations plus the density conservation equation:

$$\begin{aligned} \rho_a &= h\rho_c + (g-\epsilon)\rho_c\rho_a & (9) \\ \rho_a &= uh\rho_s + u(g-\epsilon)\rho_s\rho_a & (10) \\ \rho_a &= 1 - \rho_s - \rho_c & (11) \end{aligned}$$

where u represents the fraction of stable states which become critical when an additional grain is added.

This can be turned into an equation involving ρ_a alone, which can then be solved to give:

$$\rho_a = \frac{-1 - u - u\epsilon + ug - uh \pm \sqrt{4(g-\epsilon)hu^2 + (1 + u + u\epsilon - ug + uh)^2}}{2(g-\epsilon)u} \tag{12}$$

In the limit of small h, which is equivalent to slow drive, the zeroth order term vanishes, and after some algebra, the linear term is:

$$\rho_a = \frac{uh}{1 + u - ug + u\epsilon} \tag{13}$$

This is consistent with the conservation equation $z_{in} = z_{out} \Rightarrow h = \epsilon\rho_a$, as it must be, subject to the condition that $u = \frac{1}{g-1}$ since by conservation $\rho_a = \frac{h}{\epsilon}$. This gives:

$$\rho_c = \frac{1}{g} + \mathcal{O}(h) \tag{14}$$

$$\rho_s = \frac{g-1}{g} + \mathcal{O}(h) \tag{15}$$

Criticality occurs in the limit $\rho_a, h, \epsilon \to 0$, at the point where the order parameter ρ_a is zero and the susceptibility $\frac{\partial \rho_a}{\partial h}$ diverges.

Thus we find that in the limit of slow drive and no energy dissipation, the system exhibits signs of criticality. Both of these limits are features of the sandpile models of section 1.2. These features are also good approximations in the atmosphere: evaporation is slow compared to the timescale of condensation, and the total amount of water vapor is conserved except at cloud boundaries. It was also found that at criticality, the number of free parameters is reduced to just one which governs the relative ratio of stable states to critical states. This could correspond to the differences between different types of clouds.

1.4 Relevance to the Physics of Clouds

The physics of clouds is a complex and mathematically laborious field (see, for example, [5]), and this paper will make no attempt at a quantitative explanation. The purpose of this section is to provide qualitative evidence that the 2d Oslo model introduced in section 1.2 is a reasonable, albeit basic, model for atmospheric processes.

As explained in [2], the 2d Oslo model is meant to be roughly analogous to the process of rain droplet formation in clouds. The gradient values $z_{i,j}$ represent the (course-grained) density of water vapor at a given point in the atmosphere, and a toppling event (i.e. when $z_{i,j} > z_c$) corresponds to the condensation of water vapor into a droplet. This process occurs physically when the (local) dew-point matches the ambient temperature. When a droplet condenses, latent heat is released, which in turn causes airflow in the vicinity of the condensation event. This may then lead to further condensation events, causing a chain reaction referred to as an avalanche in the original model, or a rain event in the cloud parlance. The x-direction is meant to represent height in the atmosphere, and the y-direction allows lateral diffusion. We can understand the addition of grains at $x = 0$ as evaporation from the Earth's surface. Note that this is a slow process, whereas condensation events can happen very rapidly, thereby justifying the rule that no new grains are added until all toppling events have terminated.

The idea that atmospheric dynamics tend to evolve toward critical points shares many similarities with the meteorological idea of quasi-equilibrium. Meteorological quasi-equilibrium is encountered when the energy drive of a system occurs over much longer timescales than the dissipation, as alluded to above. Quasi-equilibrium is described as a sequence of quasi-neutral states, meaning that the system remains in the same area of phase space but never

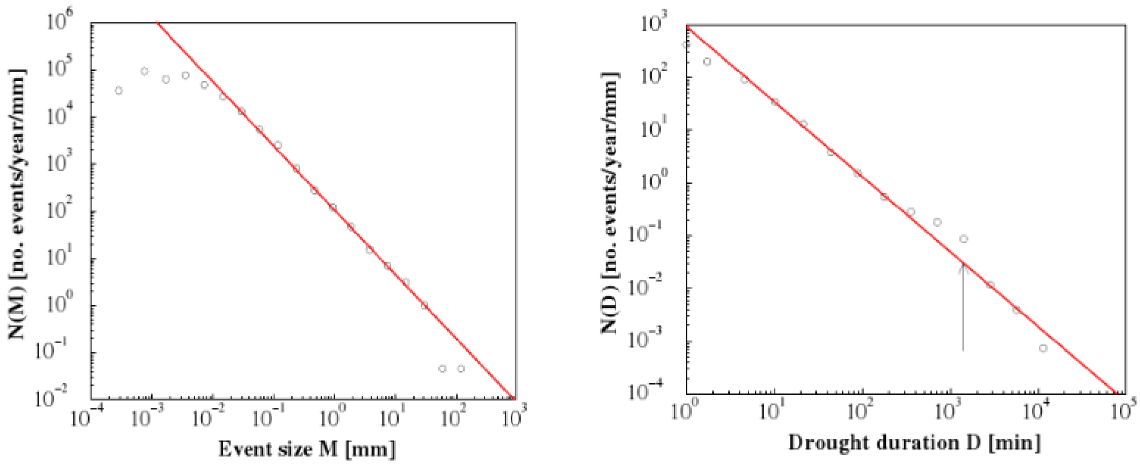

Figure 1: (*left*) Log-log plot of rain event size distribution. (*right*) Log-log plot of drought duration distribution. The arrow marks the duration of one day. The red line in each plot obeys a power law, i.e. $f(x) = x^{-\tau}$.

quite settles into full equilibrium [5]. As pointed out in [6], this is essentially the statement that the critical point of the system acts as an attractor, as assumed in SOC.

2 Methods and Results

2.1 Rainfall Data

Since the publishing of the original SOC paper, studies have observed the predicted power law behavior both in the frequency of rain events and in their relative sizes [6] [7]. Accurate collection of rain data to unprecedented precision has become possible in recent years through the use of Doppler radar and other sensitive electronic equipment. Previous data had been recorded in the form of reservoir levels recorded at regular intervals [8]. In one recent study, a Doppler radar was used to collect rainfall data for six months with a time resolution of one minute [7]. Figure 1 shows two plots taken from the data. It can be seen that the data obeys power law scaling for about three decades both for rain event size and drought length. In the drought duration plot, the arrow marks the length of a day. The typical temperature variations seen over the course of a day cause some deviation from power law behavior at that timescale.

In [6], satellite microwave retrievals of rainfall, water vapor, and other parameters, were used to illustrate that the precipitation transition behaves in a way that is mathematically similar to familiar phase transitions in physical systems. In Figure 2, it can be seen that the susceptibility, defined as the variance of the order parameter, spikes at the point of

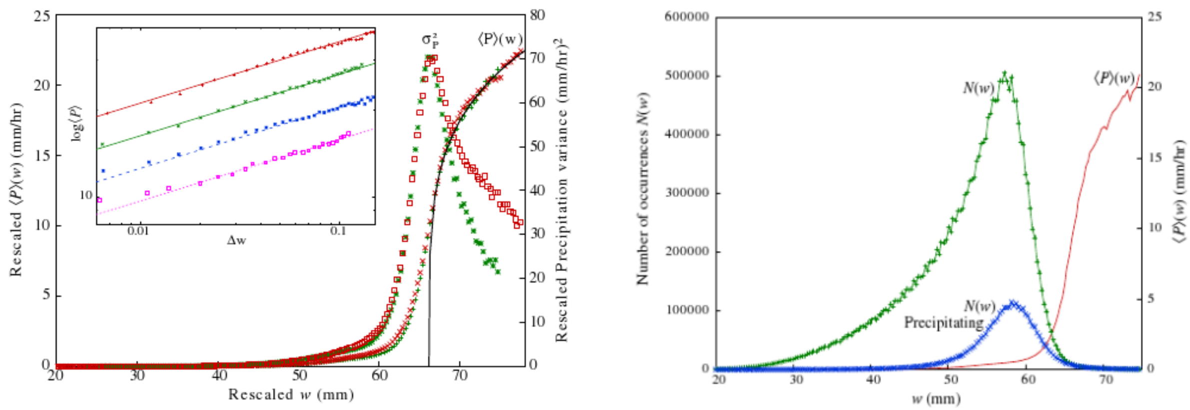

Figure 2: (*left*) Plot of susceptibility, σ_P^2, and order parameter $\langle P \rangle(w)$, versus water vapor density. The solid line is a power-law fit. The inset shows power law behavior observed in precipitation rates from different regions of the world. (*right*) Frequency plot of vapor densities during periods of drought (green) and precipitation (blue), plotted next to order parameter as a function of vapor density (red).

transition, just as is seen in typical phase transitions. The second image in Figure 2, shows that the system tends to spend most of its time right around the transition point, indicating that it is attracted to that point by the dynamics. The transition point is signified by the rapid rise of the order parameter.

2.2 Simulation Data

The simulations as described in section 1.2 were run and analyzed in [2]. Data was taken from simulations run on a 100×100 point lattice, though other lattice sizes were also run with comparable results. Figure 3 shows that the distribution of avalanche sizes obeys a power law, just as the distribution of rain event size data did. In addition, the simulations found a power law exponent of -1.4, compared to -1.36 for the collected data.

Data was also collected which shows regions of activity (i.e. avalanches) during the simulations (see Figure 4). In this model, regions of activity indicate regions where water is actively condensing. In the context of the atmosphere, these images can be interpreted as the shape of clouds as predicted by the model.

3 Conclusions

It has been shown that certain features of observational rainfall data can be approximately reproduced by means of a simple sandpile model. This agreement is especially impressive

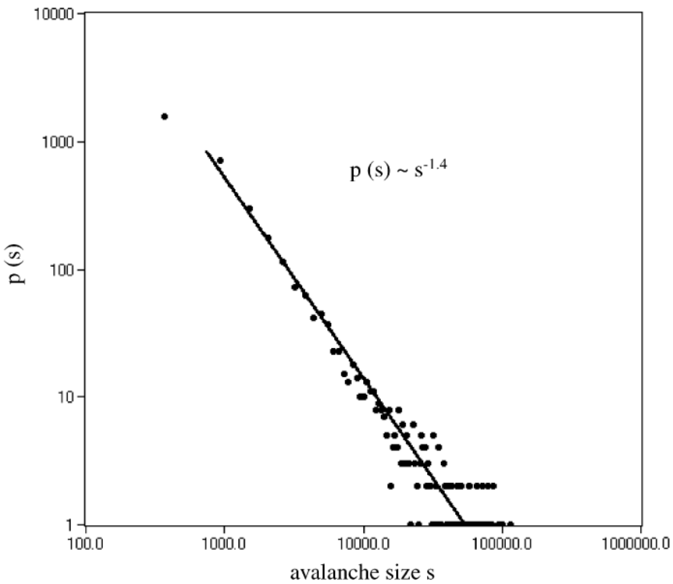

Figure 3: Frequency of avalanche sizes taken from simulation data. The black line is a power-law function, $p(s) \sim s^{-1.4}$.

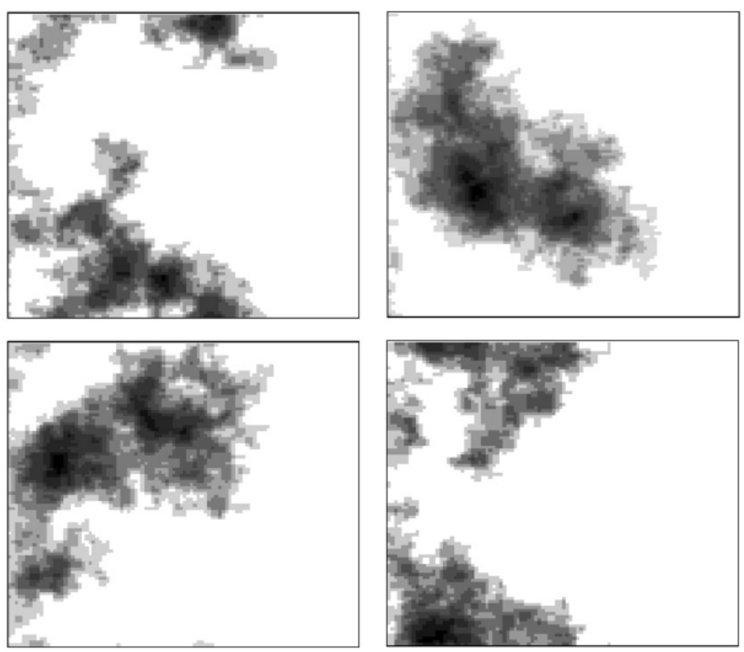

Figure 4: Regions of activity during Oslo model simulations.

considering the complexity of atmospheric processes and the simplicity of the model employed. The model also successfully generated images which closely resemble the shapes of rain clouds, indicating that perhaps cloud formation can be viewed as a sort of avalanche process. A great advantage of modeling systems with sandpile-like models is applicability to a wide array of complex systems. Sandpile models have been proposed as models for avalanches, earthquakes, and many other systems with slow drive and sudden dissipation [9].

It must be emphasized, however, that the Oslo model data is far from conclusive. The emergence of scale-invariant power law behavior does not imply that the model is capable of reproducing more complex atmospheric processes. In fact, attempts to match observational data may lead to alterations which make the model less feasible for other applications. Indeed, Dickman has argued that the latent heat produced in condensation events should inhibit condensation in surrounding areas, rather than provoke it, as assumed in the Oslo model [10]. He goes on to imply that the only way to simulate the atmosphere with any accuracy is to consider the underlying dynamics. If this is the case, then cloud physics once again becomes a complex problem in fluid dynamics, although Dickman has shown reasonable agreement with published results using vortex simulations. This creates a more realistic model at the cost of increased complexity.

In the future it would be interesting to see extensions to the Oslo model which are able to predict more complex atmospheric phenomena without considering microscopic dynamics. Updating the model to incorporate three dimensions may also be worthwhile, since emergent properties often arise only above a certain dimensionality. Significant gains could also be made with a more robust theoretical framework which applies the tools of non-equilibrium physics to the atmosphere.

Though there has been limited research at the boundary between atmospheric science and the study of emergence, the studies presented here show surprising agreement between observations and data produced by simple models. Though it seems hopelessly complex, perhaps the atmosphere can be at least partially understood in terms of symmetries and simple local interactions. As with so many unsolved problems in physics, the rewards could be enormous.

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