Spontaneous symmetry breaking shaping flowers and leaves

Reza Vafabakhsh

(December 2008)

Abstract:

Spontaneous symmetry breaking can result in emergence of complicated patterns in nature. Simple symmetry arguments can be used to describe beautiful and periodic rippled and wavy edges of flowers and leaves as well as fractal structure at the edge of a torn plastic sheet. In this paper, I will first discuss simple physical and geometrical arguments based on symmetry breaking and metrics on a curved surface that result in this pattern formation. Next, I will provide results of some theoretical and simulation studies of wrinkled pattern formation in plastic sheets and leaves.

I Introduction:

Spontaneous symmetry breaking (SSB) beautifully describes order and pattern emergence in condensed matter physics. Emergence of magnetization in ferromagnets below the Curie temperature and appearance of Goldstone bosons in models with spontaneous broken symmetry are among them. The 2008 Nobel Prize in physics went to 3 physicists for their work on spontaneous symmetry breaking in subatomic physics. In contrast to these rather abstract examples, SSB has been widely used to describe pattern formation in some more familiar phenomena. In fluid systems, SSB manifest itself in formation of stripes, spirals and waves as stable patterns and can be driven by buoyancy, surface tension or temperature. Another important example of SSB is spontaneous pattern formation in a reaction-diffusion system [1]. This is interesting because it has been argued that this process maybe related to pattern formation on animal coat and skin. As we see, SSB is widely used to describe emergence of order and pattern formation at very different levels from ferromagnets to complex patterns on an animal coat. In this paper I study another application of SSB to describe formation of complex and symmetric patterns in a wide class of leaves and flowers. As I will discuss, we do not necessary need a complex genetic code to instruct each piece of a leaf to adopt a specific shape. In fact a symmetric uniform growth rate, with a higher rate at the margins, can generate wavy and rippled patterns. I will also discuss several experiments and computer simulation that support this hypothesis.

Figure 1: Complex wavy pattern along the edge of an orchid (left) and an ornamental cabbage (right). An enhanced, uniform growth rate along the edge can generate such patters (from [2]).

II Buckling and Spontaneous Symmetry Breaking (SSB)

As discussed in the introduction, SSB can be used to describe complex pattern emergence from simple equations. SSB happens when stable solutions of an equation posse less symmetry than the equations themselves. As a simple example, if we press inward on a uniform, long plastic strip (like a ruler), the plastic strip and the force are symmetric in horizontal direction. Now by increasing the force, under this symmetric tension, ruler has two choices: either to shrink, like a spring, or to bend up or down. Simple experiments have shown the plastic strip will buckle eventually. In fact, the equations describing the elasticity of thin sheets are called Föbbl-von Karman equations. In these equations, stretching energy is linearly dependent on thickness (t) while bending energy is cubic in t [4]. So, as the sheet become thinner and thinner, stretching becomes more expensive energetically compared to bending. Therefore in a plastic sheet with a thickness of a few hundred microns, we expect to see immediate buckling under compression without a change in its length. As buckling sets in, since there is no factor to favor up or down direction, symmetry will break spontaneously and system will eventually buckle either up or down. In case of a plastic sheet, buckling is reversible and upon removal of the force, the object will snap back to its original equilibrium symmetric configuration.

Now, in another experiment, if we make a small cut in a thin plastic sheet like a wrapping sheet or a garbage bag, and tear it apart along the cut we end up with an edge which is curled up and down and highly structured (figure 2).

Figure 2: Wavy edge of a torn plastic sheet (from 2)

The equilibrium shape of the sheet consists of a cascade of waves upon waves along the edge. It turns out that even if the experiment is done in a controlled manner and with a uniform speed, the final shape of the sheet is the same. Figure 3 shows the newly formed edge at different levels of magnification. As it is seen, the images are self similar with a scaling factor of 3.2. This self similarity suggests that the edge can be considered as a fractal [2, 9]. Now the question is how uniform deformations on a flat surface can result in emergence of such a complex structure.

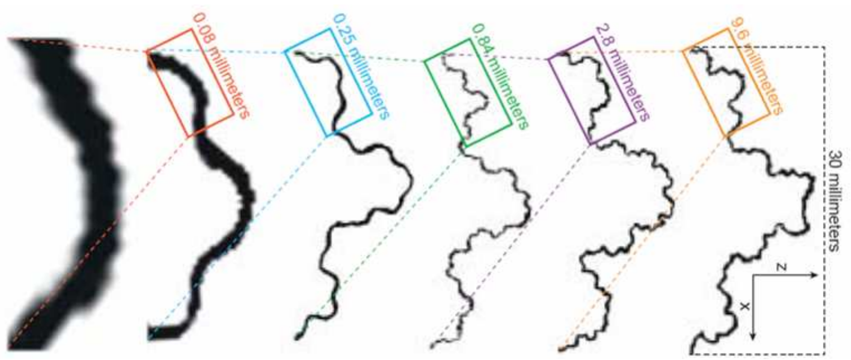

Figure 3: Successive magnification of the edge of the thin plastic sheet (thickness 0.012 mm). Each step is magnified 3.2 times with respect to the previous one (from 2).

A close look at the edge of the sheet, while being torn, shows that as the plastic is ripped, it stretches close to the edge (red line in figure 4) while remains uniform away from it (purple line in figure 4).

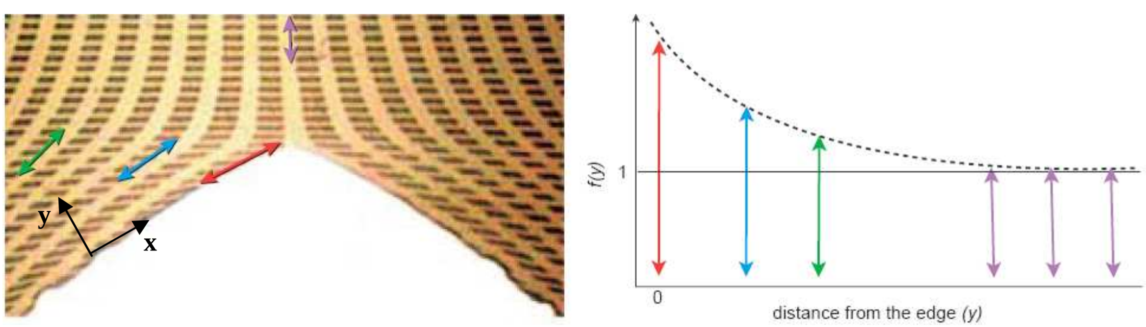

Figure 4: The distance between points along the direction of the tear changes irreversibly. This change is not uniform and is higher closer to the edge.

If the sheet were to remain flat after this irreversible deformation, it would have to rearrange all pieces of it by expanding or shrinking across the sheet. However, it is energetically more favorable for it to buckle along the edge and as it buckles up and down, it spontaneously breaks the symmetry in the vertical direction.

III Formal Description

From the mathematical point of view, having a non-uniform elongation throughout the sheet is equivalent to defining a new target metric on the surface. For the configuration shown in figure 4, the component of the metric along x-direction (g) will change and it will be a function of y ($g = g(y)$). Employing Gauss' Theorema Egregium which relates the Gaussian curvature (K) to derivatives of the metric through the following equation [5]

$K(y) = -\frac{1}{g}\frac{d^2}{dy^2}g$ we see that if g is a convex function of y, then K<0. This means that every point on the surface is a saddle-like point and in order to adopt this configuration, the sheet must buckle out of the plane spontaneously. Sharon and his collaborators did a series of systematic experiments to find the dependence of the wavelength of waves on the local properties of the sheet [6]. The two local length scales that enter are the local thickness t(y) and the geometrical local length scale $L_{geo}(y)$. This geometrical length scale is the inverse geodesic curvature along y=constant lines. The authors measured $L_{geo}(y)$ by cutting narrow strips parallel to the torn edge of the sheet and flattening them between glass coverslips. The strip curls up into a ring whose radius is $L_{geo}(y)$ (figure 5).

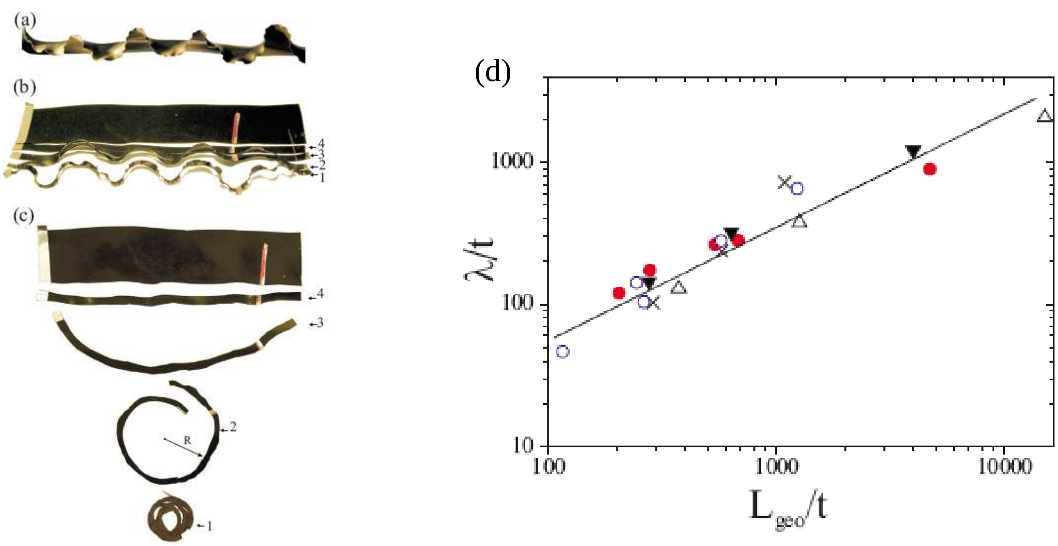

Figure 5: The process of measuring $L_{geo}(y)$. (b) Narrow strips of plastic is being cut, (c) and confined between glass plates. Radius of the formed rings is the local intrinsic curvature. (d) Normalized wavelength as a function of normalized geometric length scale collapse into a line with slope equal to 0.7 which implies $\lambda(y) = t(y)^{0.3} L_{geo}^{0.7}(y)$. Each symbol is from a sheet with different thickness (from 6).

As it is shown in figure 5d, their experiments suggest that $\lambda(y) = t(y)^{0.3} L_{geo}^{0.7}(y)$. Also, as figure 5c shows, L_{geo} decreases as $y \rightarrow 0$ which means the wavelength becomes shorter and shorter as we approach the edge. Also from the scaling behavior it is seen that if the sheet has a thickness that approaches zero, it will buckle and wrinkle everywhere along its edge which as well predicts the observation of fractal as we discussed earlier.

Marder *et al* [7, 8] and Sharon *et al* [9] presented simple discrete energy functional as $\varepsilon = \frac{\kappa}{2a}\sum_{\langle ij \rangle}\left[u_{ij}^2 - \sum_{\alpha\beta}\Delta_{ij}^{\alpha} g_{\alpha\beta}\Delta_{ij}^{\beta}\right]^2$ which upon minimization would give the shape of the surface. Here κ is energy per volume, a is a characteristic length, Δ_{ij} is the equilibrium distance between lattice points i and j and $u_{ij} = \left|\vec{u}_i - \vec{u}_j\right|$ is the distance between points when they are not in equilibrium. g is the metric tensor that they guessed and used in their algorithm. They tested this approach for a wide thin strip under compression, and observed a satisfactory agreement between the simulation and experiment result. Figure 6 shows summary of their result for 3 different metric tensors for metric along with the result of experiments on plastic sheets and beet leaves.

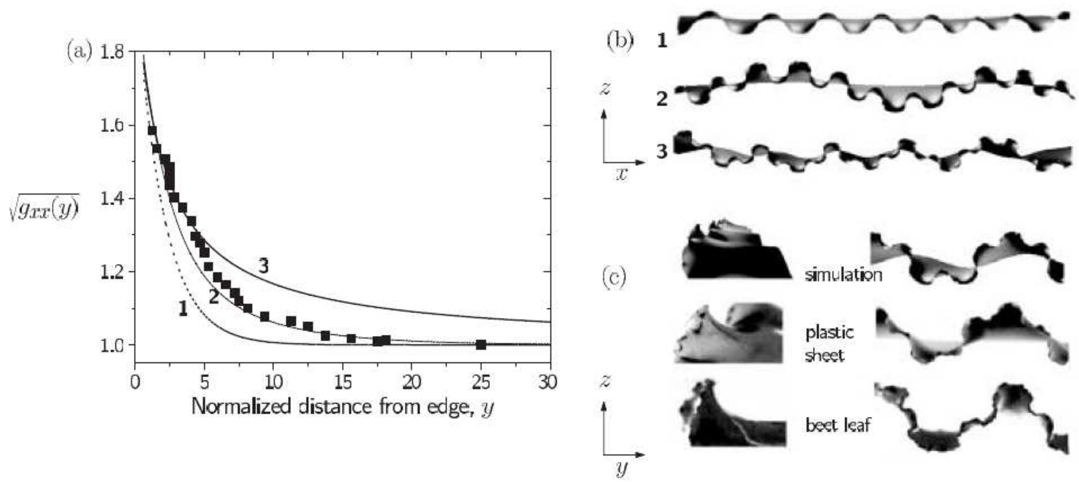

Figure 6: (a) Measured metric for a 200 micron thick plastic sheet (squares) is compared with three different trial metrics: (1) $g_{xx}(y) = \left(1 + e^{-\frac{y}{2}}\right)^2$, (2) $g_{xx}(y) = \left(1 + 0.7e^{-\frac{y}{2}} + 0.3e^{-\frac{y}{6}}\right)^2$ and (3) $g_{xx}(y) = \left(1 + \frac{2}{2+y}\right)^2$. (b) Emergent patterns as a result of minimizing the given functional for three given metrics. (c) Comparing profile of simulated sheet and experiments on plastic sheet and natural leaf.

IV Patterns in Leaves and Flowers

Apparent similarities between the edge of a torn plastic sheet and a wavy edge of a leaf inspired Sharon and co-workers to look into this problem more carefully. Earlier, in a series of experiments, Nath *et al* [6] demonstrated that precise controlling of growth rate throughout the leaf can control the overall shape of it. Therefore, interrupting the natural genetic pattern by over expressing or silencing certain genes can produce leaves with different morphological parameters. However their experiments did not differentiate the role of genetic coding and geometrical properties in shaping the leaf. In other words, does the leaf need to be programmed at each point to buckle up or down? To address this question, Sharon *et al* [2] did a simple experiment. Their motivation was to study the effect of a uniform metric change close to the leaf edge on the leaf shape. In order to do that, they applied growth regulating plant hormone auxin along the edge of eggplant leaves which are naturally smooth and flat. After a few days of treatment, they observed waviness appeared along the edges of the leaves (figure 6).

Figure 7: Growth hormone auxin is applied along the edge of a flat eggplant leaf. After 10 days of treatment, waves have emerged and they increase in amplitude later.

Sharon *et al* [5] did more experiments to test their hypothesis about similarity between pattern at the edge of a leaf and a torn plastic sheet and whether leaf waviness is a result of uniform change of metric along the leaf edge. To do so, they did the same measurement as they did on torn plastic sheets, discussed in figure 5, on natural leaves. In fact they saw that in a leaf with wavy edge, the radius of strips decrease as the edge is approached (figure 7a, b). Using this information, they could perform quantitative measurement of the metric on the leaf as a function of distance from the leaf edge (figure 7c). Repeated experiments on many different leaves all produced a linear metric for the flat part and a convex metric for the wavy part of the leaf.

The authors tried to generalize their findings and use this geometrical picture to describe formation of more complex patterns in flowers. They performed some experiments and computer simulations to study emergence of pattern in cylinders as a result of non-uniform growth toward the end [2]. Their idea was that if a cylinder grows toward one end, at some point it will break the cylindrical symmetry and buckle. They tested their hypothesis by doing experiment on tubes made of polyacrylamid gel which swells in water but shrinks in acetone. So, by inserting the tube in acetone and then dipping the end into water they can create a non-uniform but axially symmetric metric that results in a trumpet shape observed in figure 8a and simulations (figure 8b). Moreover, by making the acetone to water transition happen on a shorter distance and therefore making the metric changing steeply, they could break the cylindrical symmetry spontaneously and form a wavy edge (figure 8c). By choosing the right target metric on a cylinder, this result was observed in computer simulations as well [2, 7].

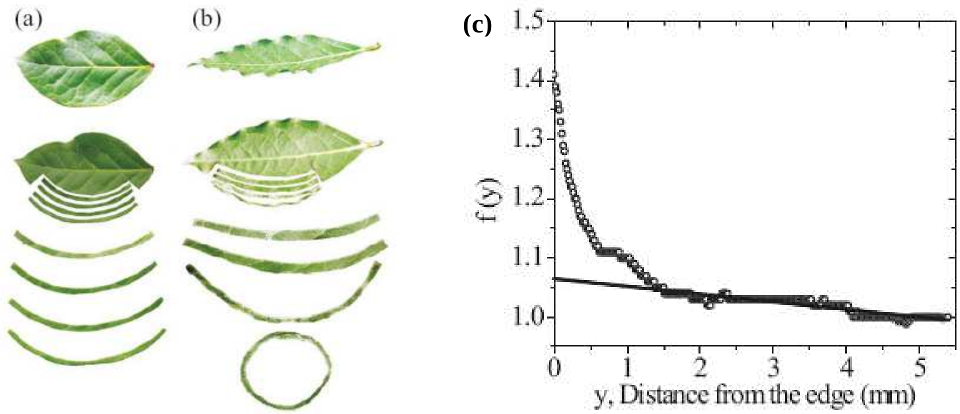

Figure 8: Intrinsic geometry of leaf studied for flat and wavy leaves with the same method discussed in figure 5. As it is seen, as leaf edge is approached $(y \to 0)$, L_{geo} (radius of curvature of thin strips) (a) for a flat leaf and (b) decreases for a wavy leaf. (c) Metric function for a wavy Wisteria leaf (circles) compared to a flat leaf with the same contour length (solid line) (from 5).

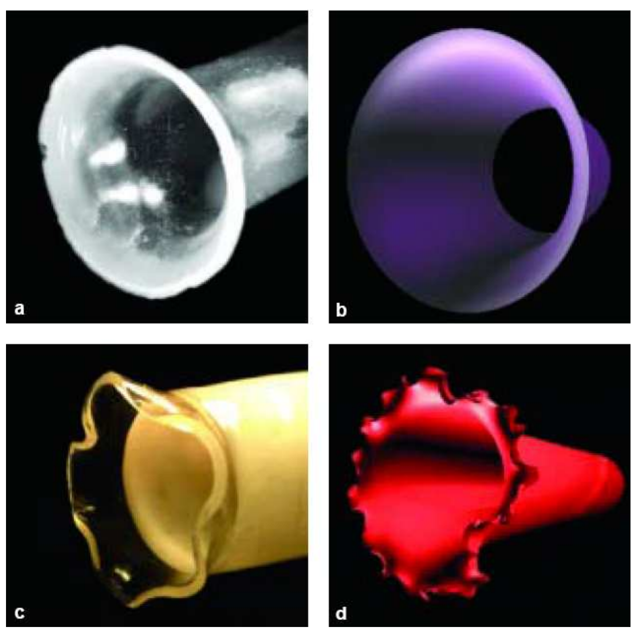

Figure 9: Experimental (a, c) and computer simulation (b, d) results for cylinders with a non-uniform growth at the edge. Buckling resulted in emergence of wavy edges.

Now this mechanism can be used to describe the formation of complex crown of the daffodil or narcissus flowers as a consequence of constant, uniform and symmetric growth of flower tissue (figure 9).

Figure 10: Formation of beautiful and complex crown of daffodil and narcissus flowers can be attributed to a uniform and symmetric growth of the tissue which has then buckled and spontaneously broken the symmetry (from 2).

V Discussion and Conclusion

In this paper I discussed formation of a wavy pattern observed in experiments on thin plastic sheets as a result of spontaneous symmetry breaking and buckling. Then I discussed how change of the target metric can be the mechanism behind this pattern formation and provided results of systematic experiments and computer simulations to verify this hypothesis. I then showed how this same mechanism can result in formation of complex patterns is leaves and flowers as well. In other words, trees and flowers don not necessarily need a complex and detailed genetic code to form a pattern. In fact, as it is shown here, simple geometry and physics arguments can describe formation of naturally occurring complex structures.

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