Oscillons formed by granular matter

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This text shortly reviews the emergence of oscillons in a layer of vibrated granular matter. First the discovery of oscillons by Umbanhowar, Melo and Swinney is reported. Second several methodically different approaches for the explanations of oscillons are given: The first and the last model approach oscillons from granular matter's properties, whereas the second model choses a more general perspective, using order parameter equations, respectively a Swift Hohenberg type of model. Finally the main reasons for the emergence of oscillons are outlined.

Contents

1 Discovery of Oscillons

Granular matter is widely found in our surroundings. Sand might only be the most popular example, but industrial sequences are also full of granulates. Therefore it has long been the aim of scientists to find rules for the behaviour of granular matter.

Looking form very far the particles in granular matter appear very small and one could be attempted to consider the whole as a continuous fluid. In some cases this seems to be not a too bad assumption, but the granular particles show another sort of interaction than molecules in fluids. What can be viewed as the common problem of scaling, which is observed in many fields. For example it is known from experience that a pile of sand does not simply flatten, like a liquid would do - at least not completely, when there is no external perturbation causing the sand to flow.

A common experiment to gain information about granular matter is to investigate a layer of it on a vibrating, horizontal plate. In 1996 Umbanhowar et al. published a paper on such a system describing a very interesting feature which they called oscillon. These oscillons are axial symmetric localized structures on the otherwise flat granular layer (Figure 1). They oscillate with half the frequency of the vibrating plate $\tilde{f} = f/2$. Where $f = \frac{\omega}{2\pi} = frac1T$ is the frequency of the vertically vibrating plate.[1] During one cycle of the plate an oscillon is a crater, in the next the granulate forms a peak and than becomes a crater again. That is why oscillons are called subharmonic or double period features. The crater state shows a rim slightly higher than the surrounding flat layer, the peak state is encircled by a slight depression.

Several oscillons, which do not have to be in the same state, were observed to coexist at the same time. So peaks and craters can be seen on the layer together. But these different states are always half a phase apart in their evolution. This can be observed as the states, peak and crater, switch at the same time. As already implied in oscillons being a period doubled phenomena oscillons switch their state during every cycle of the vibrating layer. Surprisingly those oscillons are a highly stable phenomena and usually last for several 10^5 cycles of the plate[1].

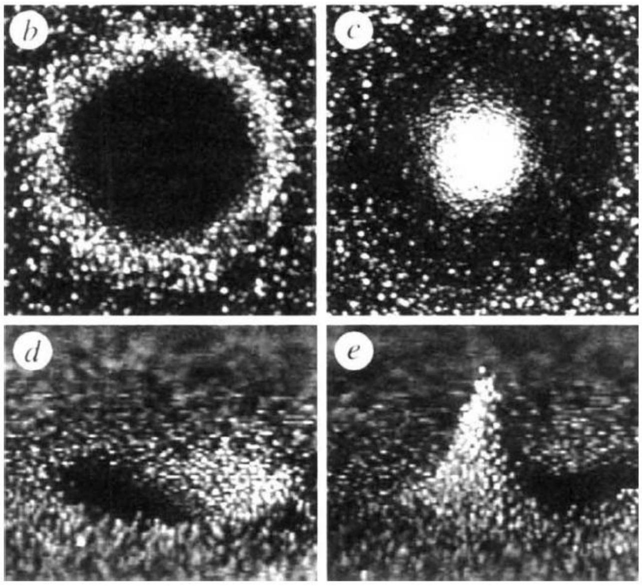

Figure 1: Oscillons of granular matter: b) top view of crater; c) top view of peak; d) side view of crater; e) side view of peak (From Umbanhowar et al. [1])

In their experiments Umbanhowar et al. used the following setup. A horizontal plate covered with a thin layer of granular matter was vibrated vertically in an upright evacuated cylinder. Oscillons were reported when the thickness of the layer was at least 14 monolayers of material, but it is not clear whether this is just due to the specific experimental setup. The vertical position of the plate can be described by $z = Asin(2\pi ft)$ where t denotes time. Therefore, when using the same granulate for all experiments, A and f are the only external parameters which can be changed. For further discussion f is kept but instead of just A the dimensionless acceleration amplitude

$\Gamma = A4\pi^2 f^2/g$, with the gravitational constant g, is introduced[1]. It can be seen that this parameter is coupled to f but can still be varied independently. This choice is common in literature and seems more general, convenient as well as appropriate than A because it directly relates to the force exerted on the granular matter, but this choice does not really change anything.

As shown in Figure 2 different patterns occur on the layer of granular matter for different points in the parameter space spanned by f and Γ.

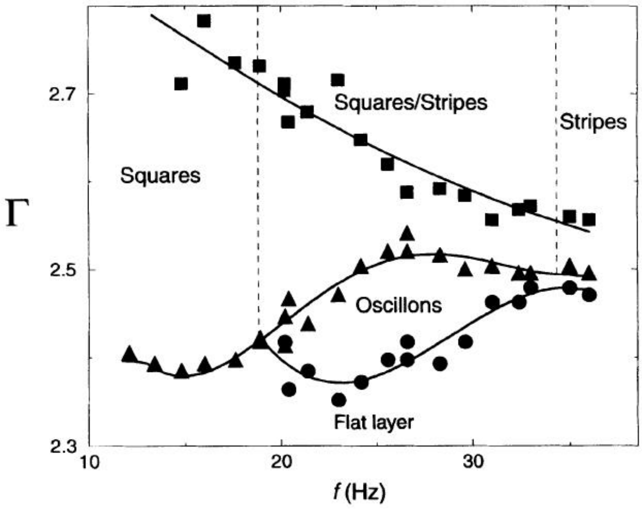

Figure 2: Different phases of a granular matter layer on a horizontal vibrating plate: Oscillons are found in the region between the flat layer and square or stripe patterns. (From Umbanhowar et al. [1])

A broader description of those patterns, which does not contain oscillons jet, can be found in an earlier publication of this group [2]. This text will focus on oscillons.

Oscillons are observed for a acceleration amplitude of $2.4 \lesssim \Gamma \lesssim 2.5$ and a driving frequency in the range of $18Hz \lesssim f \lesssim 35Hz$. From the values for Γ it can be seen that oscillons only appear far in the region were the layer takes off the plate when going up and hence collides with the plate on its way down. So during every plate cycle there is a time of free flight for the granular matter involved. The frequency region in which oscillons occur is coincident with the frequency region in which square patterns fade to stripe patterns an vice versa. But the Γ values for oscillons are below the barrier above which standing wave patterns like squares and stripes are stable. Below the lower Γ barrier for oscillons only the flat layer is stable and no patterns are observed.[1]

When approaching the oscillon region starting from a totally flat layer by keeping f constant and increasing Γ it turns out that the flat layer is also stable in the region where oscillons can occur. Only when the flat layer is perturbed oscillons can emerge. The other possibility to obtain oscillons is to approach the oscillon parameter region from a patterned state. Then the pattern will destabilize an oscillons will break away form it. That is one of the reasons why there is the idea that patterns at least in the transition region to oscillons, could be viewed as made up of oscillions[1]. The stability of a flat layer in the oscillon region of parameter space as well as the fact that when observing an oscillon the rest of the granular layer can be perfectly flat allow it to draw the following conclusion: The region in which oscillons occur is a hysteretic bifurcation region, as this allows that both, oscillons and flat layer, coexist at the same time. It is noted that the hysteresis decreases with increasing f and oscillons only occur when hysteresis is a decreasing function of frequency[1]. In other words the transition from the flat layer, respectively the flat layer with oscillons, to the parameter region where square patterns fade to stripe patterns and vice versa is of subcritical bifurcation type.

When the parameter region in which oscillons are stable is left, those either decay in 5 to 10 cycles to flat layer or they fade to another pattern. This transition to other patterns and the emergence of oscillons from patterns usually go over bound states of opposite phased oscillons. Those bound states made of opposite phased oscillons are stable. Umbanhowar et al. observed dipole like and triangular structures as well as oscillon chains. Some examples are shown in Figure 3.[1]

As a rule of thumb one can say that the interaction range of oscillons in smaller than one of their diameters and that opposite phased oscillons attract each other whereas like phased oscillons repel. The coordination number of an oscillon was always found to be smaller than four.

It was further found that the central frequency of the oscillon parameter region is proportional to

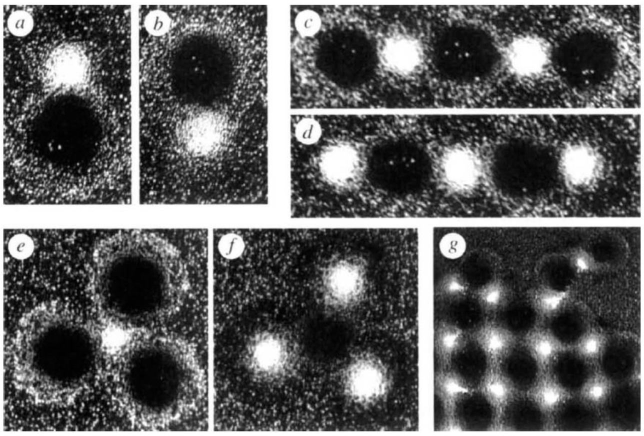

Figure 3: Some examples of bound oscillons: a,b) Dipoles; c,d) chains; e,f) triangular state with one oscillon in the middle and opposite phased oscillons surrounding it; g) bound oscillons break away from square pattern during transition due to parameter change (From Umbanhowar et al. [1])

$d^{-1/2}$ where d denotes the diameter of the granular matter used for the experiment. The kinetic energy normalized by the energy that is needed to lift a particle by one diameter is $v^2/(gd)$. v is the relative velocity to neighbouring particles Now assuming that Γ is constant and v is the maximal velocity of the plate the authors conclude that formation of oscillons in vibrated granular matter not only needs a high hysteresis but also high dissipation.[1] Hence this would till now be the two factors necessary for the occurrence of oscillons. What causes the emergence and stability and what the localization of oscillons needs further considerations.

2 Models for oscillons

Localized structures in dissipative fluid systems were investigated before oscillons, e.g. with the result that a coupling between the amplitude and frequency of wave phenomena in such systems can cause localization[4], the pinning of an interface to a periodic microscopic structure would also have the effect of localization[5]. In general nonlinear Fields were searched for stable localized solutions[6]. So it was relatively well known what could make structures localized and stable, at least all the models used had had on thing in common: A subcritical instability.

After oscillons were reported by Umbanhowar et al. different approaches were used to explain this phenomena. Some of them explicitly considering properties of granular matter[3][7][8]. Others using fully macroscopic or phenomenological models that include just a few basic characteristics which the behaviour of the system exhibits[9][10][11][12][13].

In the following three different of the many approaches to model oscillons will be introduced.

2.1 Cerda et al.

The first model introduced here was developed by Cerda, Melo and Riga. The model's idea is to separate the vibration cycle into tow different phases: The first phase is the one of free flight, it gives the major contribution to the lateral movement of granular particles. The horizontal relaxation of the layer thickness takes place during the second phase, when the layer collides with the plate.

The description of the lateral movement is implemented with the following equation which also accounts for mass conservation.

$$\begin{aligned} h(x,t_f) &= h(x,0) \\ &\quad + \int dr \left[W(r \to x)\, h(x,0) \right. \\ &\quad \left. - W(x \to r)\, h(x,0) \right] \end{aligned} \tag{1}$$

It simply implies the lateral movement with the function $W(r \to x)$. This is the probability that a column of sand moves from r to x during the first phase. Hence this equation accounts for the interaction of x with its environment in the laziest way, namely just by what one is interested - the lateral movement. In the simplest form W can be described by a δ function,

$$W(r \to x) = C\delta\left\{x - \left[r + t_f v(r,0)\right]\right\}$$

with v being the two dimensional horizontal velocity of the sand column following the relation:

$$v(x,t) = -v_0 \nabla h(x,0)$$

t_f is the time during which the lateral movement takes place. x denotes the horizontal spatial variable in two dimensions.

The thickness relaxation can than be described by a simple diffusion equation.

$$\partial_t h(x,t) = D\nabla^2 h(x,t) \quad (2)$$

Before examining the model regarding linear stability a closer look on the equations 1 and 2 can already provide some understanding about what the model will tell. The diffusion equation 2 alone would kill every perturbation, so instabilities have to be promoted by the lateral movement given by 1. On the other hand 2 is the only possibility in the model to pick the most unstable mode, respectively wavelength.

A linear stability analysis in $h(x,t) = h_0 + \xi(x,t)$ of this system gives the following results: The amplitude of the perturbing mode k grows each cycle by a factor that is basically $\sigma(k) \approx (1 - ...k^2)\, e^{\cdots k^2}$. Therefore it can be seen that a flat perturbation would be stable and there can just be one most unstable wavelength λ. It can further be concluded the phase of the perturbation changes by π each cycle. Those facts sound already very oscillon like.

The onset of the most unstable mode for $\Gamma = Ch_0 v_0 t_f / D(T - t_f)$ can than be determined for $\Gamma = 3.6$. By determining the diffusion coefficient to $D = v_0 \iota (d + \delta)$ with an unknown function ι the particle diameter d and the layer dilation δ, it turns out that $\lambda = \iota (d+\delta) \frac{v_0 t_f}{\Gamma}$. Unluckily this relation is difficult to check in experiment as it is hard to estimate δ.

To finally investigate the nonlinear regime numerical simulations were done, implemented according to the basic idea of the model with a free flight and a collision phase. It turned out that for low Γ values the flat layer is stable and in accordance to linear stability analysis gets unstable for Γ values greater than 3.6. Which is not really agreeing with the experiment, but this should also not be expected as the model is not very detailed. A square pattern could be observed in the simulations, but in contrast to the experiment the transition to this pattern is supercritical.

To obtain a subcritical transition like observed in the experiment an internal friction angle is introduced. In cases where the surface slope of the granular layer is smaller than this angle, the lateral motion of the granulate does not happen. And indeed after considering the internal friction angle subharmonic oscillon like features could be observed in numerical simulations (Figure 4). This is a hint that dissipation plays an important role in making the transition subcritical and hence also for the existence of oscillons.

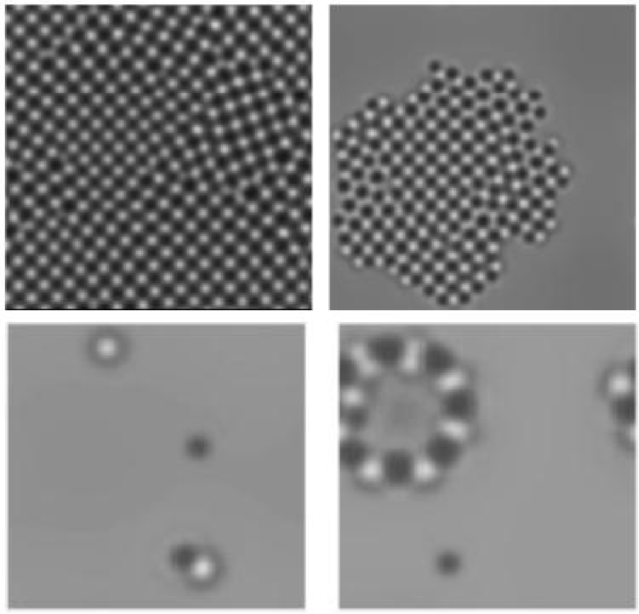

Figure 4: Simulations: Top($\Gamma = 5$, $C = 1$): left: Stable square pattern, right: Metastable squares coexisting with flat layer in subcritical region; Bottom($\Gamma = 8$, $C = 0.77$): Subharmonic features in subcritical region. (From Cerda et al. [7])

For C values near unity these oscillon are reported to be unstable, but for decreasing C and increasing f, keeping Γ constant they start to become stable. Sadly the paper does not state how the oscillon like features are obtained in the simulation. Whether an extra simulation was done or they emerge form the metastable square pattern state by separation like it was observed in the experiment (Figure 3 g))

Fact is the model obtains oscillon like features. It seems that dissipation plays an important role for the existence of oscillons. But does not become really clear why we see oscillons in exactly this parameter region, the model can just tell that they are there. The Γ dependence is at least a little bit illuminated. Because without a phase of free flight dissipation always overweights the possibilities of the driving force. Only when the granular layer recollides with the plate the forces on it are high enough to promote perturbations. On the other hand this means that dissipation stabilizes already existing structures, like the pile of sand mentioned in the introduction, so maybe also oscillons.

2.2 Aranson et al.

A further possibility to model the system, followed by Aranson and Tsimring, is an order parameter equation. This equation is a borrowing form equations used in fluid mechanics and designed based on phenomenological properties of the experiment. To take equations which are similar to those in fluid dynamics is distinct, as mentioned above granular matter is in some sense similar to a continuous fluid. Here the order parameter ψ denotes the complex amplitude of oscillations with the subharmonic frequency $\tilde{f}$. $\rho(x,t)$ is the local averaged area density of the granular matter layer, so in principle its function is similar to that of h in the model before.

$$\partial_t \psi = \gamma\psi - (1-i\omega)\,\psi + (1+ib)\,\nabla^2\psi - |\psi|^2\,\psi - \rho\,\psi \tag{3}$$

This equation is mainly constructed in a way that it exhibits all the features that lead to oscillons how they were observed in the experiment. For example it is important that the equation is invariant under $\psi \to -\psi$, so that peaks and craters are equivalent. Which is according to [12] a result of the discrete time symmetry of the system. The first term on the left hand side shall promote the excitation of standing waves from perturbations. The following two terms are determined from the dispersion relation of driven granular waves. The nonlinearity is there to cause the saturation of solutions and the last term gives the coupling to the density ρ. A second equation accounts for mass conservation in the system. The second term on the left hand side is the regular diffusion equation term, whereas the fist term is introduced as particle tend to leave regions with higher density fluctuations drifting to more calm areas.

$$\partial_t \rho = \alpha\nabla\left(\rho\nabla|\psi|^2\right) + \beta\nabla^2\rho \tag{4}$$

Obviously $\psi = 0$ and $\rho = \rho_0 = const$, in other words a flat layer, is a trivial solution for this system. This solution becomes unstable to a perturbation with wavelength λ in linear order, in two different cases: For $\tilde{\omega}b > 1+\rho_0$ the flat layer becomes unstable at $\gamma_c^2 = (\tilde{\omega}+b(1+\rho_0))^2/\left(1+b^2\right)$. For $\tilde{\omega}b < 1+\rho_0$ $\lambda \to \infty$ first gets unstable at $\gamma_c^2 = 1+\rho+\tilde{\omega}^2$.[10]

For certain parameter configuration localized axial symmetric structures, like shown in figure 5, were found. Those can be supposed to correspond to oscillons, when comparing the radial profile for ρ form figure 6 to pictures obtained from experiment.

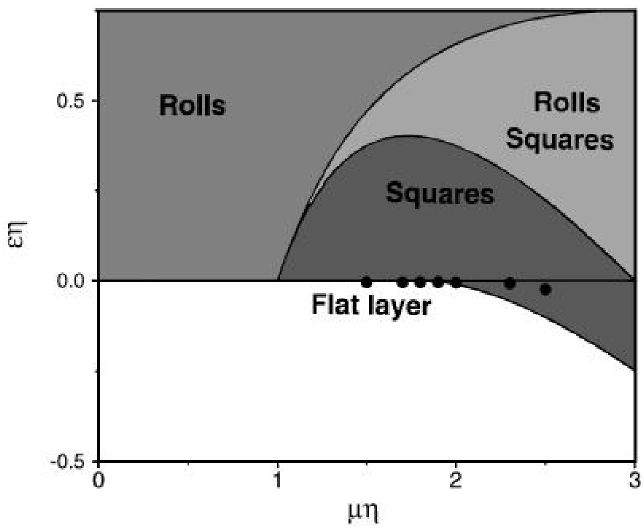

Figure 5: Phase diagram calculated by Tsimring et al.: Black dots indicate the region in which stable oscillons were observed in numerical calculations. (From Tsimring et al. [9])

Further linear stability analysis, based on axial symmetric perturbations, shows that those oscillons are, like observed in the experiment, stable for in the region of subcritical transition from the flat layer to square states. So the γ values for which oscillons are stable have to fulfil the following relation: $\gamma_{c1} < \gamma < \gamma_{c2} < \gamma_c$. At the boundaries of this parameter region the unstabilized structures begin to annihilate with others of their kind.

To investigate the behaviour of oscillons outside of the linear regime Aranson et al. apply, like Cerda et al., numerical simulations. In those nu-

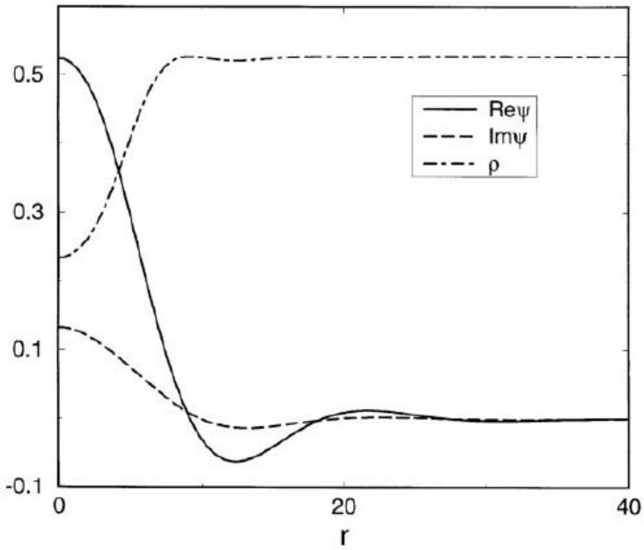

Figure 6: Radial distribution of oscillon in a crater state: (From Aranson et al. [10])

merical simulations even the interactions between oscillons could be modelled in a qualitative correct way. Dipoles are found, as well as triangular structures but no structures with a coordination number of higher than four. This is certainly a hint that the model discussed here does not only model the emergence of oscillons pretty well but goes way further. Note that also square patterns and so called roles can be described by this model. The model also tells that like phased oscillons could bind, too. But those binding strengths are in the order of "‘granular noise"’ and hence significantly weaker than bounds between opposite phased oscillons. This is also the reason why they are not observed in experiment. But those attractive interaction between like phased oscillons are also predicted by other models, for example in [12]. But this is not really surprising, as this model follows similar ideas than the one of Aranson et al.

In the numerical simulations it is found that as soon as γ goes below γ_{c1} oscillons decay very fast to the flat layer. For γ increasing above γ_{c2} it depends on the value of $\alpha/\beta\mu$, where μ is the overall average mass density, what happens. For small values, that means when the diffusion more important than the particle movement due to density fluctuations, rolls form. Those may then fade to a non ordered square pattern. For high values, hence when particle movement manly depends on local density differences the already existing oscillons nucleate more oscillons at their boundaries and form chains, exactly like it was observed in the experiment. Note that this basically means that the correct behaviour is obtained when density differences throughout the layer play an important role for the system.

This strengthens the idea that stable patterns form or are made up of oscillons. Which was already addressed by Umbanhowar et al. [1]. They had supported suggestion by adding a subharmonic frequency to the driving force of the plate. As a result the region which was formerly the transition region between squares and stripes was then filled by a hexagonal pattern, which turned into a hexagonal lattice of like phased oscillons when decreasing Γ. This process can be seen in figure 7.

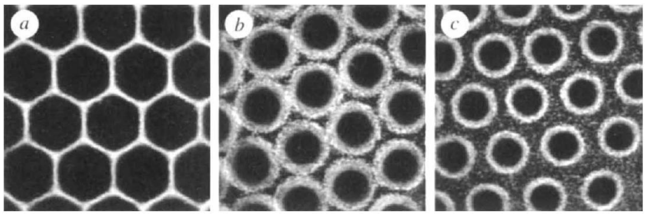

Figure 7: The transition from a hexagonal pattern to a oscillon lattice while reducing Γ in a system the has a additional subharmonic driving frequency of the plate. (From Umbanhowar et al. [1])

The advantages of this model is certainly the simplicity that nevertheless leads to the emergence of oscillons. It does not consider any particular microscopic properties of granular matter but captures the phenomenological details good enough to make predictions. This means that it can model the emergence of oscillons, or granular matter in general. But it will not give inside in what is special about granular matter that oscillons can emerge. It just models the system from what can be macroscopically observed.

2.3 Rothman

The model introduced by Rothman starts with the idea to view granular matter as an amount of non elastic colliding spheres which are viscously coupled. Then space is made continuous and locally averaged, while time stays time like we know from everyday life not like in Swift Hohenberg approaches. This leads to the following equations:

$$\partial_t^2 h = \nu \nabla^2 \partial_t h - g + B\left(h, \dot{h}, z, \dot{z}, \alpha\right) \quad (5)$$

Were ν denotes the viscosity and B is a function which describes the bouncing of the layer during layer plate collisions, $h(x,t)$ is again the layer height and z the position of the plate. When assuming the bouncing to be a linear process than B is defined by

$$\dot{h} \rightarrow \dot{z} + \alpha\left(\dot{z} - \dot{h}\right)$$

Where $\alpha(\rho)$ is a measure how good the layer adopts the velocity of the plate. It monotonically decreasing with the area density of the layer $\rho(x,t)$. The time evolution of ρ is described by a similar diffusion equation as in Aranson's model.

$$\partial_t \rho = D\left(\nabla^2 \rho - \nabla\rho \nabla h / h_0\right) \quad (6)$$

Those equations were numerically solved on a square lattice, assuming α to be some stepwise defined decreasing function[8].

To simplify comparison with the experiment a characteristic frequency f_0 and grain size were introduced in terms of D and ν. Then taking the grain size and frequency $f_0 \approx 25Hz$ used in the experiment, reasonable D and ν were determined. Simulations were started far from the stable flat state, but with constant ρ to enable pattern formation. Oscillons can be found as final solutions. Rothman notes that during the formation of the pattern the fluctuations in h result in fluctuations in ρ, so that regions of high α can emerge. This is a good explanation for the fact that oscillons switches between crater and peak state every cycle. When there is a crater at a certain spot α is high, as a result the layer there will bounce high and the region is then filled due to the second term in equation 6.[8]

From figure 8 this double period evolution of the oscillon can be seen in a nice way. And also the excitation of the oscillon is shown very clearly.

Several simulations at different points in parameter space lead to the phase diagram shown

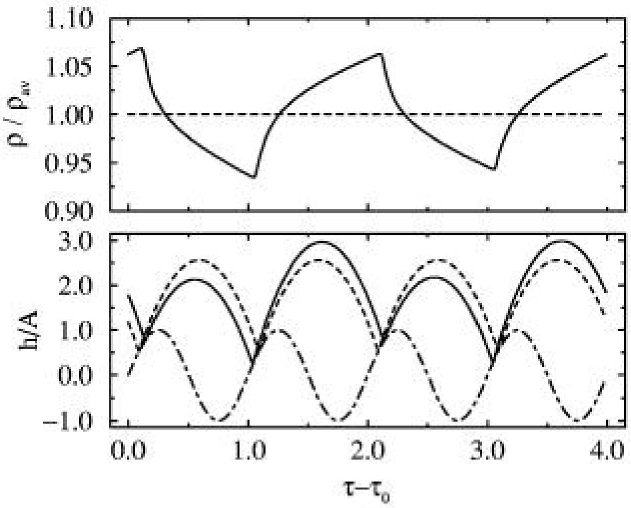

Figure 8: Evolution of ρ and h inside (solid line) and outside (dashed line) a oscillon. (Dot-dashed line: Evolution of z) (From Rothman [8])

in figure 9. From this diagram it becomes evident that the model fits the experiments pretty good, as the Γ as well as the frequency region match the experimental results fairly well. The really good Γ values are obtained because of the right choice of $\alpha \approx 0.2$. So the experimental layer is like expected pretty non bouncing, compared to a flummy for example. Instead of a square pattern this model obtains stripes but otherwise the qualitative properties of the phase diagram fit the experiment as well as the quantitative.

The radius of the oscillons, and hence also the wavelength of the most unstable mode in this parameter region, can only depend on $sqrt(D/f)$, as this is the length scale determining the mean travelling length of a particle during on plate cycle.[8]

Further insight about the region in which oscillons can be found comes from equation 6. The density fluctuations inside an oscillon should for example scale as

$$\frac{\Delta\rho_i}{\rho_0} \propto \frac{D}{a^2 f} \propto \frac{f_0}{f}$$

This means, keeping Γ constant, when f exceeds a critical value the density fluctuation in the oscillon become to small to keep it alive and it decays to a flat layer. On the other side, when f becomes to small oscillons should fade to stripes. This can be explained assuming that oscillons have similar

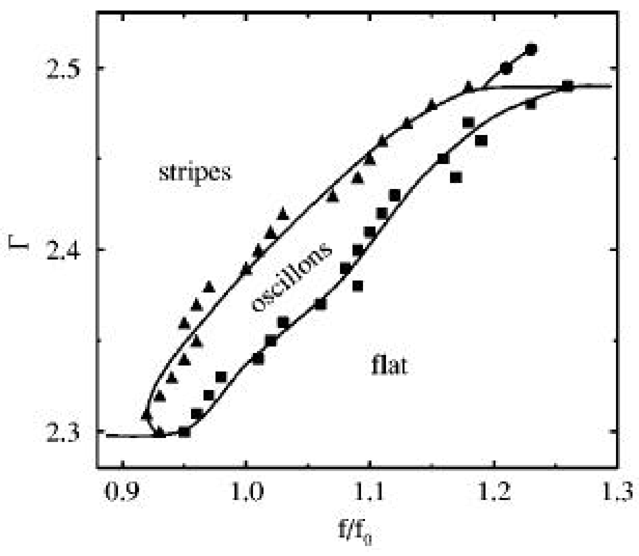

Figure 9: Phase diagram determined by simulations. (From Rothman [8])

shapes. Then the difference between the density fluctuation inside the oscillon and outside the oscillon at points each the distance l away form the oscillon border scale for small l like:

$$\frac{\Delta\rho_i - \Delta\rho_o}{\rho_0} \propto \frac{l}{R} \propto \sqrt{\frac{a^2 f}{D}} \propto \sqrt{\frac{f}{f_0}}$$

This means that for small enough f the oscillon starts to nucleate other oscillons outside its boundaries. This is the mechanism observed for the fade form oscillons to patterns in the experiment.[8]

This model also makes it plausible that opposite phased oscillons form bound states and do not annihilate. All what happens is that the state with the lower α shuffles matter into the region with the higher α and vice versa.

3 Conclusion

Already before the discovery of oscillons several ideas had been present to account for their basic features{Th88. It might have been more the need to include oscillons in the whole picture of vibrated granular matter which caused so many publications in the following time. Especially phenomenological approaches do this pretty well. Therefore the discovery of oscillons should have improved the general models existing for vibrated granular matter Especially phenomenological models, should have become more accurate, premising that those have already been on the right track.

Anyway the big question at the end is again what causes the emergence of oscillons or to put it in a more precise way: We know how they look like, why do they do so, why do they exist and why do they exist in this parameter region. I think that these are difficult questions to answer.

Several people state that hysteresis, respectively a subcritical transition from the flat layer to square patterns is necessary[12]. But, as oscillons appear in a region were the flat layer is stable when there is no perturbation and oscillons coexist with the flat layer, this is already evident from the experiment[1] one does not need a model to see this. Further this fact is necessary but that does not mean that it is sufficient. It also seems that the term subcritical does not really provide an explanation, in the common sense, because what causes the subcriticallity?

One expects that there is a promising possibility to dig deeper and understand granular matter at a scale of the size of its particles. Therefore models exceeding an phenomenological order parameter approach are needed. As mentioned above it turns out that those model are really good in the respect of describing nature. But they are missing the relation to the microscopic physical properties of granular matter. The best that can be done in this type of models is to draw parallels to already existing fluid models like e.g. Swift Hohenberg approaches and modify them according to observations made in the experiments.

It would be nice to have a link between already existing attempts[7][7] to describe granular matter from a microscopic point of view and also existing order parameter equations[10][12]. At the moment it only can be said, that simulations based on properties of granular matter tend to provide a deeper insight in the fundamental properties of granular matter, even though they do not provide analytical solutions.

From Rothman's and Cerda's papers it can for example be concluded that indeed dissipation is a main reason for the existence of oscillons, probably because of its stabilizing properties. Cerda et al. further outline why it is so important to have a phase of free flight in order to promote perturbations in the layer. This need for the free flight with a certain strength of layer plate collision afterwards sets up the lower Γ boundary for the oscillon region. Rothman further tries to explain what the reason for the oscillons' frequency region could be: For to high frequencies the temporal density fluctuations in the oscillon become to small what results in a fast decay of the oscillon. Whereas when the frequency becomes to low the oscillon starts to drive its neighbourhood so that larger patterns emerge.

Finally it should have become clear that granular matter, respectively oscillons are not as good understood as one might wish to have it. To create a fully satisfying model a lot of more research will be needed.

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