ESM 569 Term Report: Bosonisation and Fractional Statistics

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Abstract

In this essay I will justify the applicability of 1-d physics to real systems like optical lattices, carbon nanotubes, and Hall effect setups. More specifically, I will introduce the idea of bosonization to help understand the phenomenon of fractional statistics.

1. **Introduction**

Condensed matter physics has been successful in explaining and predicting effects in bulk matter (crystals, liquids, glasses, rubbers, etc.). Additionally, it has been very useful to follow developments for more imaginative numbers of dimensions. 2-dimensional phenomena for example have been the topic of much research, and have been summed under the domain of surface and interface physics. Recent discoveries have uncovered curious opportunities to explore physics for 1-dimensional systems. The examples of the fractional quantum Hall effect (FQHE) edge states, carbon nanotubes, and 1-D optical lattices will be introduced in this essay, alongside with a calculation that predicts the emergence of the collective phenomena of fractional charge and statistics.

At first, let us attempt to justify the applicability of 1-D physics to our actual 3-D world. Bulk matter can be understood in terms of 3-D physics, and 2-D physics explains surface and interface effects. It is then believable that phenomena restricted to the edges of the bulk would be a suitable domain for a 1-D description. It turns out that this is the case for FQHE edge excitations. FQHE is a two-dimensional phenomenon of excitations that can be understood roughly as small rotating domains. Domains in contact with the boundary of the region will then propagate along the edges.

In another example we can consider dilute atomic gases suspended in an optical lattice. The geometry shown below has an experimentally achievable geometry [1] and is inherently one-dimensional.

Figure 1: Schematic of an array of 1-D optical lattices.

2. **One Specific Example**

To provide space for a more detailed study, look at the electronic structure of carbon nanotubes. The dimensionality of excitations in nanotubes is due to the band structure of graphene. One way to see this is to compute the electronic dispersion that arises from the atomic p-level of carbon in the graphene sheet. The crystal lattice has a hexagonal structure, which means that there are two atoms per unit cell. If we choose the cell to have side a and orientation along the $\hat{y}$-axis, then we can approximate the multiparticle electron state using the single atomic eigenfunctions ψ_p:

$$\begin{aligned}\psi_p^+(r) &= \frac{1}{\sqrt{2N}}\sum_R e^{ik\cdot R}\left[\phi_p(r-R)+\phi_p(r-a\hat{y}-R)\right]\\ \psi_p^-(r) &= \frac{1}{\sqrt{2N}}\sum_R e^{ik\cdot R}\left[\phi_p(r-R)-\phi_p(r-a\hat{y}-R)\right].\end{aligned} \tag{1}$$

The two different wavefunctions correspond to symmetric and antisymmetric combinations of the atomic wavefunctions in the unit cell. The sum is taken over all Bravais lattice points. In the tight binding approximation [6],

$$\begin{aligned}&\epsilon_p^\pm(k) = \left\langle \psi_p^\pm \left| H \right| \psi_p^\pm \right\rangle = \\ &= \left[\frac{1}{2N}\sum_j e^{-ik\cdot r_j}\left(\langle r_j| \pm \langle r_j + a\hat{y}|\right)\right]\left[\sum_{m,<i>} t\left(c_{r_m}^\dagger c_{r_m - r_i} + h.c.\right)\right]\left[\sum_{j'} e^{ik\cdot r_{j'}}\left(|r_{j'}\rangle \pm |r_{j'} + a\hat{y}\rangle\right)\right],\end{aligned} \tag{2}$$

where the terms i are only the closest neighbors, and the constant t is a measure of the overlap of neighboring wavefunctions.

$$\begin{aligned}\epsilon_p^\pm(k) = \frac{1}{2N}\sum_{jj'mi} e^{-ik\cdot(r_j - r_{j'})}[&\langle r_j|t\left(c_{r_m}^\dagger c_{r_m-r_i}+h.c.\right)|r_{j'}\rangle \pm \langle r_j|t\left(c_{r_m}^\dagger c_{r_m-r_i}+h.c.\right)|r_{j'}+a\hat{y}\rangle + \\ &\pm\langle r_j + a\hat{y}|t\left(c_{r_m}^\dagger c_{r_m-r_i}+h.c.\right)|r_{j'}\rangle + \langle r_j + a\hat{y}|t\left(c_{r_m}^\dagger c_{r_m-r_i}+h.c.\right)|r_{j'}+a\hat{y}\rangle]\end{aligned} \tag{3}$$

There are six terms in each summation corresponding to the six principal directions in the lattice. Writing out the sums and contracting the field operators with the bras and the kets will give δ-functions for r_j and $r_{j'}$. Keeping only nearest neighbors and simplifying takes us to [7]

$$\epsilon_p^\pm(k) = \pm|t|\sqrt{1+4cos\left(\frac{3k_y a}{2}\right)cos\left(\frac{\sqrt{3}k_x a}{2}\right)+4cos^2\left(\frac{\sqrt{3}k_x a}{2}\right)}. \tag{4}$$

Notice that the symmetric and antisymmetric combinations give rise to two different bands that are plotted below.

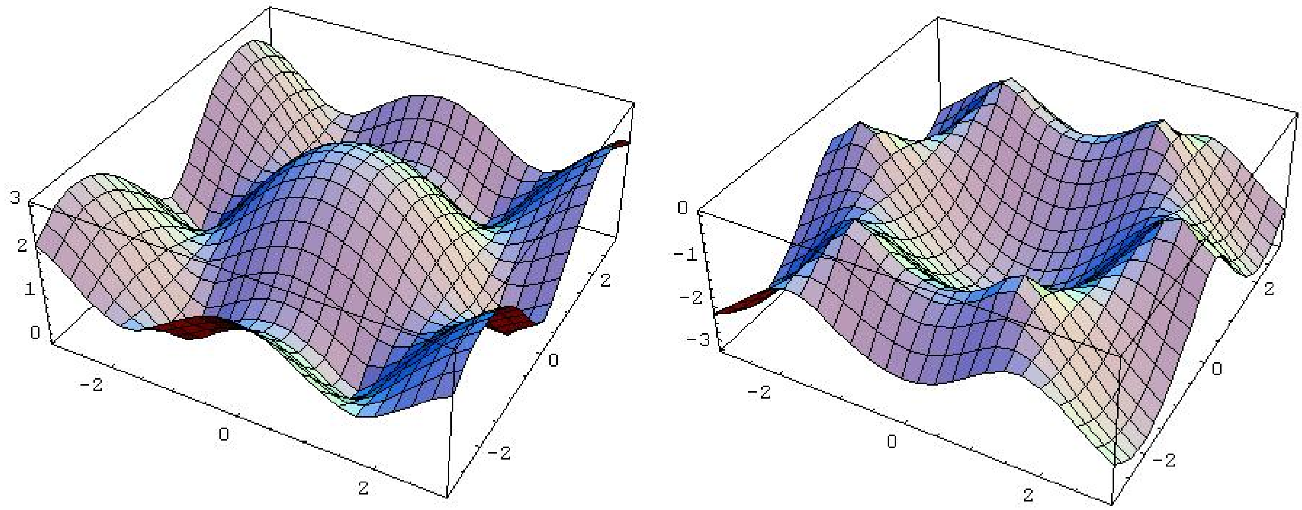

Figure 2: (Left) Symmetric configuration band. (Right) Antisymmetric configuration band.

The two manifolds touch in six distinct points in the first Brilloin zone. In the ground state p-electrons inhabit the lower band and the upper band is empty. If a graphene sheet is seamed in a way that the axial direction of the tube corresponds to a direction that includes two of the points of contact, then the nanotube will be conducting. Also, if the conduction electrons inhabit only the area near those contact points, then the momentum of the electrons is constrained in strictly one dimension. This statement validates a 1-D approach to the problem.

3. **Bosonization**

With this concrete example we can move on to introduce a theory that can handle the interactions of electrons in one dimension. One peculiarity of 1D is large strength of the iteractions. In a handwaving sort of argument we can claim that a particle cannot avoid a collision with an oncoming particle, because there is no space to step aside. Interactions are by no means small, so they cannot be treated as a perturbation around a free theory. One way to attack this problem is a substitution of variables called bosonization, which is directly applicable to FQHE edge states and carbon nanotubes. As a consequence of this technique we will see the emergence of a quasiparticle with an exotic commutation relation. For this first part of the development follow Ref.[2], Chapter 3.

Begin with the character of the single electron dispersion relation. Starting with (4) it is straightforward to show that for small energies the conducting electron dispersion relation is linear. This is also visible from Fig.1: the two bands are very near cones at the points of contact. We can write down the following free Hamiltonian:

$$\mathcal{H}_{free} = \sum_{k,r=R,L} v_F(\epsilon_r k - k_F) c^\dagger_{r,k} c_{r,k}. \tag{5}$$

L and R stand for right- and left-moving particles; $\epsilon_L = -1$, $\epsilon_R = 1$. The electronic density fluctuation ρ can be written in terms of a superposition of particle-hole excitations:

$$\rho^\dagger(q) = \sum_k c^\dagger_{k+q} c_k. \tag{6}$$

The essence of bosonization is rewriting the Hamiltonian in terms of a particle-hole pair (phonon). The pair consists of two fermions and is therefore a boson. Define the bosonic operators:

$$\begin{aligned} b^\dagger_p &= \left(\frac{2\pi}{L|p|}\right)^{\frac{1}{2}} \sum_r \Theta(\epsilon_r p) \rho^\dagger_r(p) \\ b_p &= \left(\frac{2\pi}{L|p|}\right)^{\frac{1}{2}} \sum_r \Theta(\epsilon_r p) \rho^\dagger_r(-p), \end{aligned} \tag{7}$$

where Θ is the Heavyside step function. Take the commutator of the boson operator with the Hamiltonian.

$$\begin{aligned} [b_{p_0}, \mathcal{H}] &= \left(\frac{2\pi}{L|p|}\right)^{\frac{1}{2}} \sum_{r,k} \left[\rho^\dagger_R(-p_0), v_F(\epsilon_r k - k_F) c^\dagger_{r,k} c_{r,k}\right] = \\ &= \left(\frac{2\pi}{L|p|}\right)^{\frac{1}{2}} \sum_{k,k_1} v_F(k - k_F)(c^\dagger_{R,k_1-p_0} c_{R,k} \delta_{k_1,k} - c^\dagger_{R,k} c_{R,k_1} \delta_{k_1-p_0,k}) = \\ &= \left(\frac{2\pi}{L|p|}\right)^{\frac{1}{2}} \sum_k v_F p_0 c^\dagger_{R,k-p_0} c_{R,k} = \\ &= v_F p_0 b_{p_0} \end{aligned} \tag{8}$$

The Hamiltonian

$$\mathcal{H}_{free} = \sum_p v_F |p| b_p^\dagger b_p \tag{9}$$

satisfies the above commutator. Therefore, if we assume that the boson operators generate a complete basis, then this must be the right way to express the Hamiltonian in b and $b^\dagger$. The importance of this expression is the fact that the free Hamiltonian is quadratic (not quartic, as one might expect) in the density fluctuations.
Consider the interacting part:

$$\mathcal{H}_{int} = \frac{1}{2}\sum \int dx dx' V(x-x')\rho(x)\rho(x'). \tag{10}$$

In momentum space,

$$\begin{aligned}\mathcal{H}_{int} =& \frac{1}{2L}\sum_{k,k',q} V q c_{k+q}^\dagger c_{k'-q}^\dagger c_{k'} c_k = \\ =& \frac{1}{2L}\sum_q V(q)\rho(q)\rho(-q).\end{aligned} \tag{11}$$

So both $\mathcal{H}_{free}$ and $\mathcal{H}_{int}$ are quadratic in the bosonic operators. Next, introduce the fields $\phi(x)$ and $\theta(x)$ defined by the equations

$$\begin{aligned}\partial_x\phi(x) =& -\pi[\rho_R(x)+\rho_L(x)] \\ \partial_x\theta(x) =& \pi[\rho_R(x)-\rho_L(x)],\end{aligned} \tag{12}$$

Note $\rho = \rho_L + \rho_R$, so Eq.(9) and Eq.(11) promise that we can expand $\mathcal{H}_{free}$ in terms of $\partial\phi\partial\theta$, $(\partial\theta)^2$, and $(\partial\phi)^2$. However, inversion symmetry tells us that the Hamiltonian stays invariant under $x \to -x$. We have

$$\begin{aligned}\rho(x) \to& \tilde{\rho}(-x) \\ \partial_x\psi(x) \to& \partial_x\tilde{\psi}(-x) \\ \Rightarrow \partial_x\phi(x) =& \partial_x\tilde{\phi}(-x) \\ \partial_x\theta(x) =& -\partial_x\tilde{\theta}(-x)\end{aligned} \tag{13}$$

Now, since $\partial_x \theta$ changes signs and $\partial_x \phi$ doesn't, then the Hamiltonian can be completely described by the expression

$$\mathcal{H} = \frac{v}{2\pi}\left[g(\partial_x\theta)^2 + \frac{1}{g}(\partial_x\phi)^2\right]. \tag{14}$$

g is a dimensionless parameter, and v is a velocity, and both quantities depend on the interaction strength. Going back to the left- and right-moving density description,

$$\mathcal{H} = \frac{v\pi}{2}\left[g(\rho_R - \rho_L)^2 + \frac{1}{g}(\rho_R + \rho_L)^2\right], \tag{15}$$

and substituting again

$$\begin{aligned} n_R \equiv & \frac{1}{2\pi}\partial_x\theta_R \equiv \frac{1}{2}[g(\rho_R - \rho_L) - (\rho_R + \rho_L)] \\ n_L \equiv & -\frac{1}{2\pi}\partial_x\theta_L \equiv -\frac{1}{2}[g(\rho_R - \rho_L) + (\rho_R + \rho_L)], \end{aligned} \tag{16}$$

we obtain the expression

$$\mathcal{H} = \frac{v\pi}{g}(n_R^2 + n_L^2). \tag{17}$$

This innocent-looking diagonalized form implies that irrelevant of the interaction strength, we can rewrite the Hamiltonian in terms of free field excitations. These excitations have linear dispersion and a velocity depending on the electron interaction strength. Introduce the quasiparticle fields

$$\begin{aligned} \psi_R^\dagger(x) \equiv & \rho_0^{\frac{1}{2}} e^{-i\theta_R(x)} \\ \psi_L^\dagger(x) \equiv & \rho_0^{\frac{1}{2}} e^{-i\theta_L(x)}. \end{aligned} \tag{18}$$

$$\mathcal{H} = \frac{v\pi}{g}\left(\partial_x\psi_R^\dagger\partial_x\psi_R + \partial_x\psi_L^\dagger\partial_x\psi_L\right) \tag{19}$$

In this form the Hamiltonian can be interpreted as the sum of kinetic energies of right- and left-moving noninteracting quasiparticles. Each of these particles is a superposition of strongly interacting electrons, but at this level we can disregard the underlying structure and only deal with the effective Hamiltonian.

One important aspect of the quasiparticle is its commutation relation, which becomes apparent in this statement [3]:

$$e^{i\theta_R(x)}e^{i\theta_R(x')} = e^{i\pi g\ sgn(x-x')}e^{i\theta_R(x')}e^{i\theta_R(x)}, \tag{20}$$

where we have used the Baker-Hausdorff formula and the commutator

$$\begin{aligned}[\theta_R(x), \theta_R(x')] =&[g\theta(x) + \phi(x), g\theta(x') + \phi(x')] = \\ =&g[\phi(x), \theta(x')] + g[\theta(x), \phi(x')] = \\ =&i\pi g\ sgn(x - x').\end{aligned} \tag{21}$$

We have assumed that θ and ϕ commute with themselves. The last line will not be justified completely here, but can be seen to make sense: in (12) we see that $\partial_x\phi$ gives the total density amplitude, and (18) associates θ with the phase. Amplitude and phase are then canonically conjugate:

$$[\theta(x), \phi(x')] = \frac{i\pi}{2} sgn(x - x') \tag{22}$$

Now (20) tells us something very interesting. Commuting the quasiparticle ψ field results in picking up a general phase $e^{i\pi g}$, which is referred to as fractional or braiding statistics. The fact that none of the underlying components has this property confirms that the quasiparticle is essentially a collective phenomenon.

4. **Experimental verification**

Various interference experiments have been proposed to observe braiding statistics [4]. Experimental data has been found to be compatible with the theory on several occasions. However, only recently there has been conclusive evidence for experimental confirmation [5]. In that setup the authors litographically define an interferomteter that uses tunneling of edge states to a 2-D quantum Hall effect potential hill. The hill is of a size small enough that it acts as a quantum dot. That enables the measurement of interference fringes as tunneling conductance oscillations as a function of the magnetic flux through the dot. These fringes meet the predictions for collective particles of fractional statistics.

One of the main reasons to pursue experimental advance in measuring braiding statistics is the notion that such states could be used for quantum information processing.

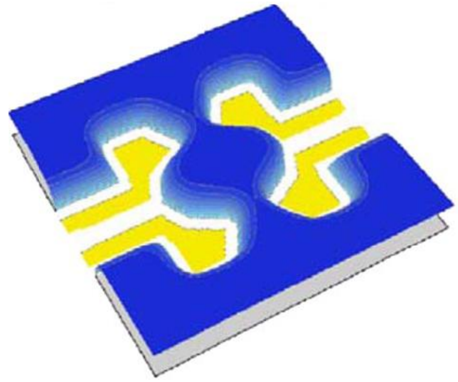

Figure 3: Quasiparticle interferometer device. Tunneling occurs at the saddle points in the twoconstrictions.

5. **Conclusion**

In this essay we have introduced the idea of 1-D physics, and we have defended its applicability to phenomena in carbon nanotubes. We have also introduced some ideas from the bosonisation technique, only as much as to justify the existence of one trait (fractional statistics) of emergent behavior. The discussion is by no means complete or rigorous, but intends to provide some flavor for the rich phenomenology of 1-D. We complete the circle with an example for an experimental measurement of that phenomenon.

6. **References**

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