Through Kaleidoscope Eyes
Spin Glasses Experimental Results and Theoretical Concepts

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December 11, 2007

Abstract

A spin glass describes a system of spins on a lattice (or a crystal) where the interactions are frustrated as well as disordered. Studying such systems presents many theoretical difficulties, but over the last twenty years considerable advances have been made. In this paper I will discuss some of the experimental results that motivated theorists and some of the concepts that were developed to deal with the physics of disordered system. Specifically, the Edwards-Anderson model and their formulation the the relevant order parameters will be discussed as well as the Sherrington-Kirkpatrick model which gave a mean field, exactly soluble version of Edwards-Anderson model. I will also talk about Parisi's solution and the non-trivial ergodicity breaking that occurs in spin glasses. Spin glasses and the physics of disordered systems have found a wide range of applicability and it is the goal to review some of the experimental features and theoretical concepts developed.

1 Introduction

Spin glasses first came to the attention of physicists in the mid- to late 1960's through the study of magnetic impurities in non-magnetic materials. The name was first coined by Bryan Coles when he was studying magnetic ions Mn or Fe on non-magnetic host metals Cu or Au. The term "glass" was used to indicate two observed facts about these systems: (1) The magnetic ions tended to "freeze in" but without any particular ordering (so it was like an amorphous solid such as glass) and (2) The low temperature specific heat was observed to be linear in T, a feature that is present in conventional glasses.

In the early 1970's that more accurate experiments were able to measure a cusp in the magnetic susceptibility when external magnetic fields were kept small. This had to be a phase transition of some sort, and it was Edwards and Anderson who produced a paper in 1975 that introduced a convenient and simple physical picture. They established that the two important factors leading to spin glass behavior were: **(1)** Competition between the different interactions amongst the magnetic moments so that no single configuration is uniquely favored (frustration) and **(2)** The interactions must be at least partially random (disordered).

From this basic frame work, a number of theoretical questions arise: (1) The "ordered" phase is characterized by an order parameter. What is the order parameter for spin glasses? (2) The "ordered" phase generally has a lower symmetry than the corresponding disordered phase. What is the broken symmetry? (3) How does one do statistical mechanics for a system with structural disorder? (4) If the relaxation time for the system is long, then the system is inherently non-ergodic. What is the proper way to treat ergodicity breaking?

In this brief report, I will review some of the experimental results that were found and how this motivated Edwards and Anderson to propose their model for spin glasses. I will also discuss the Sherrington-Kirkpartick model, an exactly soluble mean field simplification of the Edwards-Anderson model which led to insights about ergodicity and replica symmetry breaking. Since the mid-1970's many of these techniques presented have made their way into other fields. It is a goal of this report to give a general overview of these methods and motivations.

2 Experimental Results

In this section, I will briefly review these experimental results and their implications for theory. There are essentially three experimental results that is characteristic of all spin glasses (1) Frozen in disorder, (2) lack of periodic long range order and (3) remanence magnetic fields. There are other effects such as hysteresis, but these depended strongly on the type of material [6].

2.1 Frozen-in Disorder

The fact that there is no long range order but "frozen-in" disorder (i.e. the system relaxes to a disordered state below some temperature T_f) and no long range order means that one should expect the average magnetization $\vec{m} = \frac{1}{N}\sum_i \vec{m}_i$ to be zero, but the *local* magnetization, $m_i = \langle s_i \rangle$ should be non-zero.

Now, anti-ferromagnetic ordering also has a vanishing average magnetization, but this case can be ruled out through neutron scattering experiments which should find Bragg peaks if there was a periodic spin structure. But in a large system, how does one measure the susceptibility of a single site?

The answer is that the the spontaneous local magnetization will decrease the expected susceptibility from the usual Curie-Weiss law. To see this, recall that the single site susceptibility is defined as the amount of magnetization induced at a site, i by an external magnetic field acting *only* at the site. Call this field, h_i. The *local* susceptibility is defined as:

$$\chi_{ii} = \frac{\partial m_i}{\partial h_i} \tag{1}$$

Now, the fluctuation-correlation theorem relates the susceptibility to the correlation functions by:

$$k_B T \chi_{ii} = \langle (S_i - \langle S_i \rangle)^2 \rangle = 1 - m_i^2 \tag{2}$$

In obtaining the last equality, I used that $S_i^2 = 1$. Now, averaging over all sites relates this quantity to the average susceptibility that one would calculate in say the Ising model. One finds:

$$\chi_{\rm loc} = \frac{1}{N}\sum_i \chi_{ii} = \frac{1 - \frac{1}{N}\sum_i m_i^2}{k_B T} \tag{3}$$

Now, the Curie-Weiss law says that $\chi \propto \frac{1}{T}$ and so comparing with the above expression, one finds that the average susceptibility deviates from the standard $1/T$ behavior in a way that depends on the local spontaneous magnetization. In general, it is difficult experimentally to apply a magnetic field at a single site. Rather, people measure the uniform susceptibility. For an approxmately Gaussian random distribution of spin interactions, it can be shown that the off diagonal elements are vanishingly small and the dominant contribution is from the local magnetization, $\chi \sim \chi_{\rm loc}$ [6]. If there is a singular behavior in $\chi_{\rm loc}$, then there should also be a singular behvaior in χ. This singularity can be seen in Figure 1. Now the *local* spontaneous magnetization explains the cusp in the susceptibility curves, but one can see that the critical temperature T_f depends on the frequency of the applied magnetic field. It turns out that this cusp like behavior occurs for a wide frequency range (see Figure 2). This is in contrast to a conventional magnetic where there is no significant frequency dependence below the characteristic microscopic frequencies of the system.

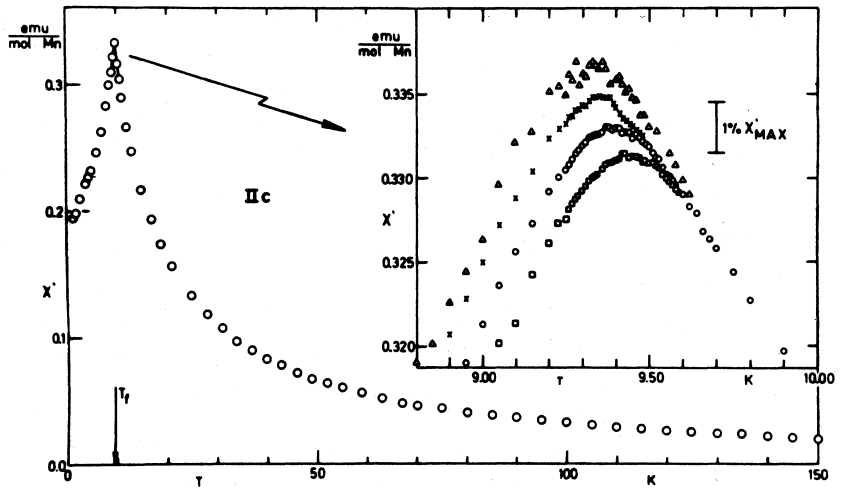

Figure 1: Singularity in the magnetic susceptibility. The data points are for various frequencies of the magnetic field [10]

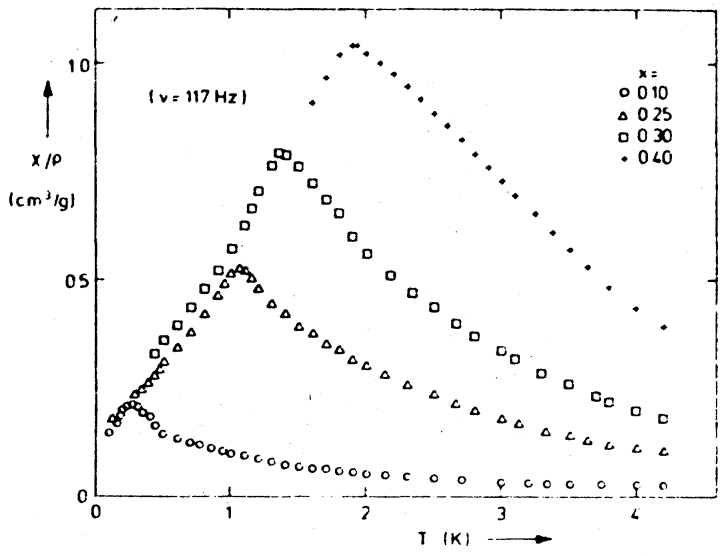

Figure 2: Differeing frequencies give different critical temperatures [4].

2.2 Remanence Effects Below T_f

Another important feature of all spin glasses is the onset of remanence effects below T_f. As shown in the figure, the behavior of the susceptibility depends strongly on how the experiment is done. Applying the field and then cooling the sample in this field ("field cooling") results in a larger susceptibility while cooling the sample down in zero field ("zero field cooling") and then applying the magnetic field results in a lower susceptibility. More importantly, they found experimentally that the effect was reversible, meaning they were able to go up in down in temperature and measure the susceptibility and obtain the same value (the susceptibility is independent of history). This is indicated in by the arrows in Figure 3.

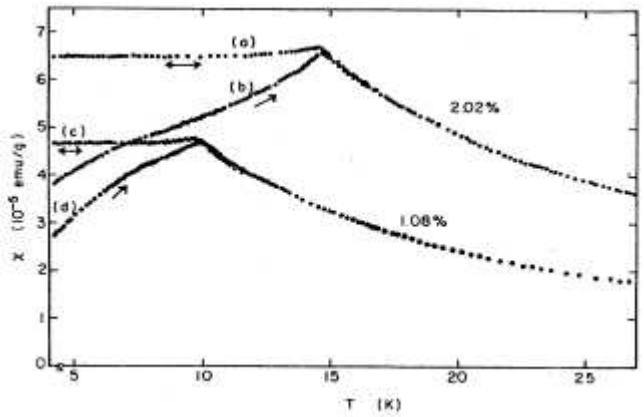

Figure 3: The remanence effects. The behavior of the low temperature susceptibility depends on how one cools the material [11]

2.3 Experimental Conclusions

From these types of experiments, physicists at the time came to conclusion that these systems must have many metastable states below T_f. There are many *roughly equal* spin configurations which the system can pick out and the exact one depends on the details of the experiment like the frequency of the applied magnetic field, the speed at which the sample is cooled or whether or not the sample is cooled in a magnetic field. The picture phase space that people had in mind in the mid-1970's was one where there were many possible ground states separated by finite potential wells.

3 Edwards-Anderson Spin Glass

Because spin glasses appear in such a wide variety of different materials, it suggested that there might be a simple model that could describe a spin glass material if the *essential* features were somehow modeled. Edwards and Anderson surmised correctly that what was essential for spin glass phenomena was (1) competing interactions (frustration) and (2) randomness [15].

The phenomenological model they developed was one based on experiments. They observed that there was a lack of long range order and thus chose the interactions to be Gaussian random with mean zero for simplicity and created a *random bond* model where the interactions between spins are chosen probabilistically.

$$H = -\frac{1}{2}\sum_{\langle i,j\rangle} J_{ij}\vec{S}_i \cdot \vec{S}_j \tag{4}$$

Here J_{ij} represents the interaction between neighboring sites i, j and is a random variable with a Gaussian distribution. In addition to this, they postulated that the frequency dependence and the remanence magnetic fields could be explained by the presence of many energy minima. When the system cools down past T_f the spins notice that there is a minimum in the energy and settle down to this state. This is a fairly simple physical picture, but it presents several difficulties. To make any predictions, Edwards and Anderson had to introduce new techniques to deal with random interactions and characterize an order parameter without long range order [15].

3.1 Random Interactions: Annealed vs. Quenched Disorder and Self Averaging

In the systems of interest, one placed magnetic impurities onto a substrate that is in general non-magnetic and it is these impurities which create the random and frustrated interactions. These random variables themselves may fluctuate with time. For instance, the atoms may take some time to diffuse through the material on a time scale τ_{dis} [3].

The other important time scale is the observation time, τ_{exp}. When $\tau_{exp} \gg \tau_{dis}$ the random variables come to thermal equilibrium first. In the partition function, one can replace the random interaction by the average equilibrium value. This is called *annealed* disorder. The free energy is then

$$F = -k_B T \log\left([Z]_{av}\right)$$

The other extreme is when $\tau_{dis} \gg \tau_{exp}$. This is called *quenced* disorder and is different from annealed disorder in that the impurities have not reached their equilibrium configurations. This situation is like that of salad dressing. If one shakes the dressing, on a short enough time scale, the oil drops will be distributed through the vinegar with some probability distribution,

but on a long enough time scale, the oil and vinegar will separate. Clearly the statistical distribution used depends on the time scale [14].

The question is, what quantity *is* close to the equilibrium value and not fluctuating wildly? It turns out that the quantity is the free energy. Let the energy of each subsystem by $E+\delta E$ where δE is the sample-to-sample fluctuation. Putting all the samples together, the energy is extensive and so scales as N and one expects the fluctuations to scale as $\sqrt{N}$. The relative fluctuation should then scale as $\delta E/E \sim N^{-1/2}$ which goes to zero as the system size goes to infinity. Such a quantity is called *self averaging*. Implicit in this is that the *interactions are short ranged*. Since the fluctuations go to zero, one can make a prediction for E and thus a theory should be centered around such a quantity [6].

In summary, one finds that for spin glasses, average over interactions can not be done; magnetic impurities are not in equilibrium. Instead, if one considers a quantity (generally extensive) that is self averaging then one can calculate the average of such a quantity over many samples. For a large enough system, the fluctuations in this quantity will then be small and a predictive theory can be made.The relevant self-averaging quantity is the free energy and one averages the free energy over many samples [3].

$$[F]_{av} = -k_B T[\log(Z)]_{av} \tag{5}$$

3.2 The Replica Trick

The trick is based on the identity $\log(Z) = \lim_{n\to 0} \frac{Z^n-1}{n}$ which comes from noting the Taylor expansion of $Z^n \sim 1 + n\log(Z) + \ldots$. about $n \sim 0$. Hence,

$$[F]_{av} = -k_B T[\log(Z)] = \lim_{n\to 0} \frac{[Z^n]_{av} - 1}{n} \tag{6}$$

The average now is over a partition function which has been replicated n-times. This is much easier to compute than averaging a logarithm. If the interactions have a Gaussian distribution, evaluating $[\cdot]_{av}$ amounts to doing Gaussian integrals.

One can then perform $[\cdot]_{av}$ by *averaging* over the interactions as in the annealed case. The different replicas do not interact with each other and hence, Z^n is easily computed. The Hamiltonian is simply a sum of Hamiltonians for each individual copy.

$$H_{ea}\{S^\alpha\} = -\frac{1}{2}\sum_{ij} J_{ij} S_i^\alpha S_j^\alpha - H\sum_i S_i^{a,z} \tag{7}$$

Now, it is clear that such a Hamiltonian is symmetric with interchanging any of the replicas with each other, but it is not clear that this symmetry remains intact as one continues to $n \sim 0$. This is the source of replica symmetry breaking which I discuss later.

3.3 New Order Parameters

The first important to task is to find an order parameter which will allow one to distinguish between the two different states. Edwards and Anderson first proposed an order parameter that looked at the order parameter as something dynamical. Then, they formulate this in terms of thermodynamic variables by thinking of different time slices as replicas.

The first type of order parameter is much more physical. Edwards and Anderson considered the situation where an experimenter measures a spin at one moment and after an infinitely long time, measures the spin again. If there is a local spontaneous magnetization, then this quantity will be non zero [3, 5, 6].

$$q_{ea} = \lim_{\tau\to\infty}\lim_{N\to\infty} q(\tau)\,; \qquad q(\tau) = \frac{1}{N}\sum_i [\langle S_i(0)\cdot S_i(\tau)\rangle_t]_{av} \tag{8}$$

This can be assessed probabilistically [5]. They find an order parameter that has the right singularity structure, namely it has different behavior above and below T_f. This is encouraging, but the problem with this order parameter is that it is dynamical and its computation in a microscopic theory is hard; it does not use the machinery of equilibrium statistical mechanics [15].

Instead they note that the above situation is similar to the replica theory only that the copies occur in time. Within the replica theory formulation of the free energy, they propose the order parameter (the replica overlap)

$$q^{ab} = \langle S_i^a S_i^b\rangle \qquad a \neq b \quad a, b \text{ replica indices} \tag{9}$$

If one passes through the transition, the spin at i will be frozen in and thus q will be non-zero. Above the transition, the spins in different replicas (subsystems) are uncorrelated and so q will vanish. Utilizing the quenched free energy, they are able to calculate the susceptibility that has the cusp behavior seen in experiment [5]. χ_c is the Curie-Weiss law, $\chi_c \sim 1/T$.

$$\chi = \chi_c\left[1 - \left(\frac{T_c}{T}\right)^2\right] \tag{10}$$

In Edwards-Anderson's paper they essentially assumed that these two were the same, namely, the replica overlap q^{ab} is given by a replica independent order parameter q_{EA}. It will be seen later that this is not the case.

4 Sherrington-Kirkpatrick Model: Mean Field Spin Glass

The results of the Edwards-Anderson calculation were certainly promising. With random interactions and the replica trick, one was able to predict a cusp behavior in the susceptibility

but to arrive at this result, a number of approximations had to be made to make computing the integrals tractable. Sherrington and Kirkpatrick were motivated by this to develop a theory in which an exact solution is possible [15]. To do this, they drew analogies to the mean field Ising model.

The Ising model is soluble in the thermodynamic limit if (1) The spins interact equally with one another (they interact with a mean field at each location, hence the interactions are *long ranged*) and (2) the exchange scales inversely with the number of spins (to have a finite energy when the thermodynamic limit is taken) [6]. This lead to the Sherrington-Kirkpatrick Hamiltonian which has a Hamiltonian similar to the EA model but whose interactions come from a distribution with a mean and variance that scales inversely with the number of spins.

$$\mathcal{H} = -\sum_{ij} J_{ij} S_i S_j - h \sum_i S_i \qquad P(J_{ij}) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(J_{ij}-\mu)^2/(2\sigma^2} \tag{11}$$

where $P(J_{ij})$ is the distribution from which the interactions are drawn from and $\mu = J_0/N$ and $\sigma^2 = J^2/N$ constants. They solve this by using the replica trick and replicating the Hamiltonian n times. By the Hubbard-Stratantovich (integrating the Gaussian backwards) they obtain a quadratic model in which the exponential terms are proportional to N. Now, *if* one is allowed to exchanged the thermodynamic limit with the replica limit $n \to 0$, then the exponential is sharply peaked and a steepest descent approximation can be done [6]. It is believed that this order of limits is not really allowed and is the source of the error in the SK solution [1].

In this way, they obtain an effective partition function which they solve via the saddle point methods at the expense of introducing self-consistency equations that have to be solved. They then choose the simplest possible ansatz which is the EA ansatz, namely,

$$y^{ab} = \lim_{n\to 0}[\langle S^a S^b \rangle]_{av} = q^{ab} = q_{ea} \qquad x^a = \lim_{n\to 0}[\langle S^a \rangle]_{av} = m \qquad \text{for all } a \tag{12}$$

Essentially, they guess the solution that y^{ab} and x^a are independent of their replica indices. This yielded the cusp in the susceptibility for a set of external parameters. However, Almeida and Thouless showed that this solution is in fact is an unstable point for low temperatures and in fact leads a negative entropy solution which is unphysical [1].

4.1 Non-trivial Ergodicity Breaking

The SK mean field solution gives an unphysical result and the naturally one wants to know why? It turns out that the answer lies in the ansatz used. Typically, when one takes the thermodynamic limit one is assuming that the energy barrier is approximately proportional the length of the boundary L^{d-1}. The probability of jumping to another valley is simply given by the Arrhenius relation. In the limit as $N \to \infty$, one expects that the activation energy to jump into a different microstate is infinitely large and so the system is trapped in a potential minimum [6].

For the case of spin glasses, the situation is much more complicated. For a spin glass it turns out that there are many metastable states (see Figure 4). By taking the thermodynamics limit, one restricts the system to a valley but the valley has many metastable states.Thus, there are contributions to the quantity q^{ab} that come from the probability of being in different sub-valleys. [6].

Contrast this with the definition of q_{EA}. In q_{EA} one asks the question of what is the probability that a spin is in a state s at time t and at a later time $t+\tau$. In other words, the system picks a well with probability P_a and a spin configuration s in that well. In q^{ab} it is possible to have replica a in one sub-valley and replica b in another sub-valley. The system picks a well with probability P_a with some spin configuration s_a and another well with probability P_b with some spin configuration s_b. Thus a *joint* probability of the two wells enters. Notice too that this can not be cured by a symmetry breaking field h. This only breaks the degeneracy $s \to -s$ but *not* the degeneracy in possible metastable states. The two clearly are not be the same in the case of metastable sub-valleys [3].

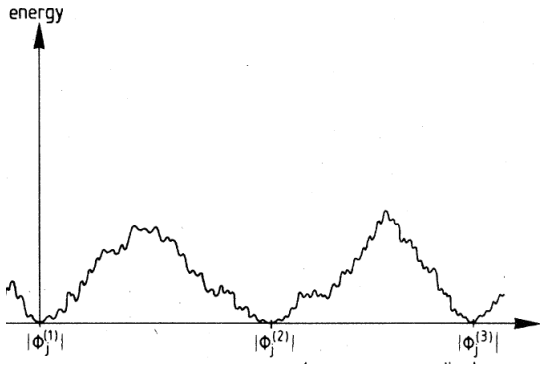

Figure 4: Cartoon phase space of a spin glass. There are many degenerate minima possible whose activation energy does not scale with the system size. The thermodynamic limit does not break ergodicity in the usual sense (restricting to a *single* phase space coordinate). Rather it breaks ergodicity down to a *subset* of metastable phase space coordinates. [3]

4.2 The Parisi Solution

Parisi had the insight that the thermodynamic limit placed one in a valley, but that this valley had many degenerate minima. One should not have a single order parameter, but rather a set (potentially infinite) of order parameters. He viewed q^{ab} the replica overlap as an $n \times n$ matrix with zeros on the diagonal (or 1, it is simply a constant factor) [13,15]. Near the critical temperature, he then applied Landau theory to the problem. The symmetry of the problem is the replica symmetry: changing the labelling of the replicas should not changed the replicated free energy. [13]. Expanding in powers of the order parameter, he

obtained the effective free energy

$$F[q] = \lim_{n\to 0} \frac{1}{2n}\left[\theta \mathrm{tr}\left(q^2\right) - \frac{1}{3}\mathrm{tr}\left(q^3\right) - \frac{1}{6}\sum_{ab}(q^{ab})^4\right] \tag{13}$$

Here, the trace preserves the replica symmetry (which amounts to permuting indices). There are many possible fourth order terms, but the one Parisi choose to keep is the one shown above [13]. Other terms would only add small corrections [6, 16].

Parisi then considered dividing up this matrix into an integral number of smaller intervals and assigns variational parameters q_i as follows: take the $n \times n$ and assign q_0 to every entry of the matrix. Now divide this matrix into $n/m_1 \cdot n/m_1$ number of sub blocks of size m_1. In the off diagonal blocks nothing is changed while in the diagonal blocks all the q_0's are changed to q_1's. One then does this for each of the sub blocks along the diagonal (see Figure 5). Each of the q_i then is a possible order parameter for the system. The system $q^{\alpha,\beta}$ can be

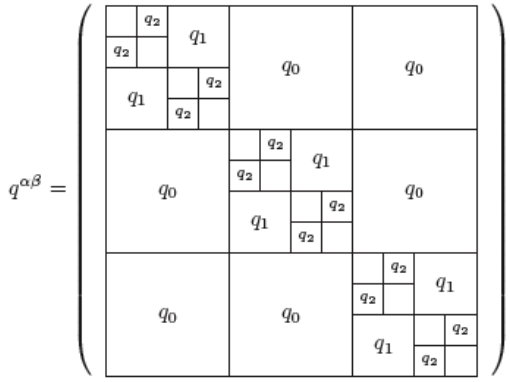

Figure 5: Matrix representation of the Parisi ansatz. The blocks represent the value of the order parameter [15]

in a block with values q_0 or in a block with value q_1. In this way, Parisi achieved an infinite set of order parameters. Replica symmetry breaking, then, is the inequivalence of each of these values [3, 6].

The Parisi solution for the order parameter matrix can also be written down as a tree (see Figure 6). The circles represent the n replicas, the vertical distance represents the value of q and q^{ab} is where the branches from a, b meet [3]. In the limit where this is done an infinite number of times, one has $m_i \to x, x \in (0, 1)$, and one can then think of the q_i as $q(x)$ a function. For different parameters of phase space, Parisi found stable solutions [13].

1. $q(x) = q = q_{EA}$; replica symmetric solution
2. $q(x) = q_0$ for $0 \leq\leq x_1$ and $q(x) = q_1$ for $x_2 \leq x \leq 1$ and monotonically increasing between x_1 and x_2. ; replica symmetry breaking solution

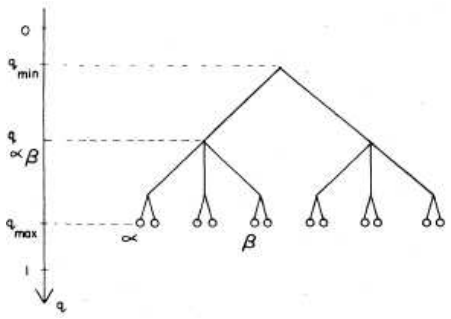

Figure 6: Replica symmetry breaking scheme. The circles represent the n replicas. To find the order parameter q^{ab} simply follow the branches of the tree from a and b until they meet. The intersection is the value of their order parameter. [3]

For a certain temperature, Parisi found a solution where all the order parameters are the same. This is the replica symmetric solution that Sherrington-Kirkpatrick and Edwards-Anderson found ($q^{ab} = q_{EA}$). Below the critical temperature, the other solution appears and one finds that all the order parameters do not have to be the same. This corresponds to replica symmetry breaking. [3, 6, 15]. With his anstaz, Parisi was able to obtain the SK solution for a certain set of parameters ($q(x) = q_{EA}$) and at the same extend the result to the low temperature regime where the SK ansatz gave unphysical results (negative entropy). The picture for a spin glass is that above the critical temperature the valleys in phase space do not have a wealth of degenerate minima. Hence the SK solution is valid. Below a critical temperature, there are many degenerate minima and the SK solution is no longer valid; there are many degenerate minima. This is the source of glassy behavior below T_f such as the remanence effects and frequency dependence found in experiment.

5 Future Directions

Thus far only the mean field model has been discussed. It is clearly not a realistic situation, and as with the Ising model, one would like to move away from mean field theory. Exactly how to do this is still unknown and is the main research thrust in spin glass research. Indeed, it is still a mystery as to whether or not Parisi's solution is an artifact of the mean field replica approach. Fisher and Huse have proposed a droplet picture based on percolating spins from droplets of aligned spins [7,8]. In this case there is no replica symmetry breaking solution, but it is still open to debate whether their methods are valid. Personally the interesting questions are SLE (percolation) descriptions of spin glasses. Several recent works have explored the use of SLE's in solving the short range spin glass problem [2, 9].

However, physicists are not boring enough to study the same problem for thirty years. Disordered systems and systems with competing interactions occur in many different areas of science and it is no surprise that many of the techniques and ideas outlined here have appeared in other fields. People have applied the study of spin glasses to optimization

problems in mathematics, neural networks, immunology and even error correcting codes [12]. In either case, spin glasses are still an interesting theoretical problem.

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