Stock market crashes

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Abstract

This paper begins with introducing the concept of crash in stock market. Different spatial and temporal scales in stock markets are to be discussed. In order to clearly manifest that crashes are emergent state, the concept of drawdowns is introduced as a superior way to the traditional returns of price. Afterwards, the mechanisms leading to positive feedbacks, such as imitative behavior and herding between investors are reviewed. A simplified spin model is constructed to make the picture easy to capture. Based upon preceding preparation, as a mathematical model, a phenomenological approach to stock market crashes is constructed.

1 Introduction

The total world market capitalization rose from \$3.38 trillion in 1983 to \$38.7 trillion in 1999. No doubt nowadays stock market investment plays a big role in the global economy and directly or indirectly relates to individuals' life. A stock market crash, a huge disaster, occurring simultaneously on most of the stock markets of the world as witnessed in October 1987 would amount to the quasi-instantaneous evaporation of trillions of dollars. Presently stock market crashes are under enthusiastic study because not only it is of manifesto practical value but also they exemplify the class of phenomena known as "extreme events", which is common to various "complex systems".

A central property of a complex system is the possible occurrence of coherent large-scale collective behaviors with a very rich structure, resulting from the repeated nonlinear interactions among its constituents. Stock market is an exemplary system with hierarchy of various scales of interacting networks. Spatially, the whole world stock market is composed of several major, mutually interacting stock markets, and each market is running by numerous closely or loosely related individual investors. Temporarily, stock markets can increase steadily over years, or have some outstanding changes during a week, or even crash within a day. As to almost all complex systems, predicting the detailed evolution of stock markets is hopeless and meaningless. However, it does not exclude the more interesting possibility of predicting phases of evolution of stock markets that really count, like crashes. Such extreme events express more than anything else the underlying "forces" usually hidden by almost perfect balance and thus provide the potential for a better scientific understanding of the system.

Unfortunately, since stock market crashes are momentous financial events, most approaches to explain crashes search for possible mechanisms or effects that operate at very short time scales. Meanwhile a radically different view [1] is also reasonably proposed: the underlying cause of the crash must be searched months and years before it, in the progressive increasing build-up of market cooperativity or effective interactions between investors, often translated into accelerating ascent of the market price (the bubble). According to this "critical" point of view, a crash has fundamentally an endogenous origin, a systemic instability, which is constructed progressively by the market as a whole, as a self-organizing process, and, on the other hand the triggering factors, as a kind of fluctuation causing a needle standing on its ends to fall, are secondary. So the main task is to understand how such large-scale patterns of catastrophic nature might evolve from a series of interactions on the smallest and increasingly larger scales.

2 From returns of price to drawdrowns

It is believed that crashes are different from normal market days. To determine whether the crashes are just extreme variations or totally anomalous requires a definite way to qualify them. A traditional way is to look at the distribution of daily returns (the daily change of the indices). Figure 1 shows the distribution of daily returns of the DJIA and of the Nasdaq index for the period January 2nd, 1990 till September 29, 2000. The lines shown in Figure 1 correspond to represent the data by an exponential function. It can be seen that the normal market days with mild daily returns obey the lines well, while large returns

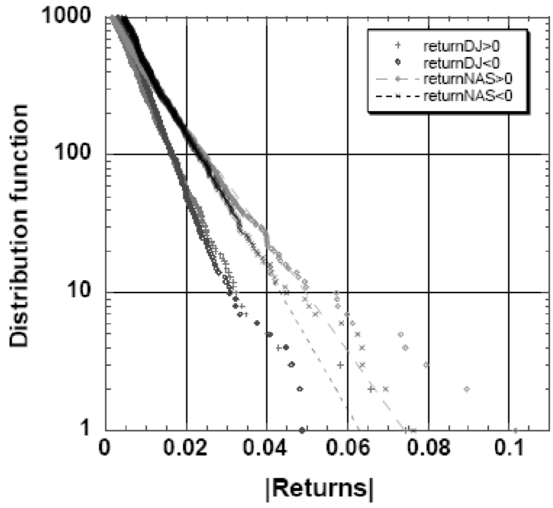

Figure 1: Distribution of daily returns for the DJIA and the Nasdaq index for the period Jan. 2nd, 1990 till Sept. 29, 2000. The lines corresponds to fits of the data by an exponential law. The branches of negative returns have been folded back onto the positive returns for comparison. Reproduced from [1]

deviate conspicuously. The upward convexity of the trajectories defined by the symbols for the Nasdaq qualifies a stretched exponential model [2] which embodies the fact that the tail of the distribution is "fatter", meaning there is higher possibility and larger risks of large drops. It can be calculated from the exponential model, which assumes daily returns independent, that the probability to observe a return amplitude larger than 10 standard deviations is 0.000045, which corresponds to 1 event in 22,026 days, or in 88 years, and the crash-like drop of 22.6% of October 19, 1987 would correspond to one event in 5.20 million years, which qualifies it as an "outlier". Thus, according to the exponential model, a 10% return amplitude does not qualify as an "outlier", in a clear-cut and undisputable manner. But the above estimate is got by looking at a fixed frequency, returns of a day. As the frequency varied, for example from a day to a week, the distribution may change, sometime violently in quantity. So the discrimination between normal and abnormal returns depends on the choice for the frequency distribution. It is related to the time scale used. Qualifying what is the correct description of the frequency distribution, especially for large positive and negative returns, is a delicate problem that is still a hot domain for research. Due to the lack of certainty on the best choice for the frequency distribution, this approach does not seem the most adequate for characterizing anomalous events.

Another crucial problem of daily returns is that it washes out the correlation between successive days by assuming independent returns. It is probably true for normal days when the temporal correlation decays too quickly to survive to the next day. But strong correlations may appear at special times precisely characterized by the occurrence of extreme events. Consider, for instance, the 14 largest drawdowns that have occurred in the Dow

rank	starting time	index value	duration (days)	loss
1	1987.786	2508.16	4	−30.7%
2	1914.579	76.7	2	−28.8%
3	1929.818	301.22	3	−23.6%
4	1933.549	108.67	4	−18.6%
5	1932.249	77.15	8	−18.5%
6	1929.852	238.19	4	−16.6%
7	1929.835	273.51	2	−16.6%
8	1932.630	67.5	1	−14.8%
9	1931.93	90.14	7	−14.3%
10	1932.694	76.54	3	−13.9%
11	1974.719	674.05	11	−13.3%
12	1930.444	239.69	4	−12.4%
13	1931.735	109.86	5	−12.9%
14	1998.649	8602.65	4	−12.4%

Table 1: Characteristics of the 14 largest drawdowns of the Dow Jones Industrial Average in last century. The starting dates are given in decimal years. Reproduced from [1]

Jones Industrial Average in last century. Their characteristics are presented in table 1. Only 3 lasted one or two days, whereas 9 lasted four days or more. It suggests the existence of a transient correlation when large successive drops are observed. If the common approach is followed to examine probability of the daily distribution of returns in order to estimate a crash of 30% occurring over three days with three successive losses of exactly 10%, assuming independence, it is the probability of one daily loss of 10% times the probability of one daily loss of 10% times the probability of one daily loss of 10%, giving 10^{-9}. This corresponds to a 1 event in 1 billion trading days. This utterly impossible result relies on the incorrect hypothesis that these three events are independent. Simply looking at daily returns and at their distributions has destroyed the information that the daily returns may be correlated at special times.

A better way is to analyze drawdowns. A drawdown is defined as a persistent decrease in the price over consecutive days. A drawdown is thus the cumulative loss from the last maximum to the next minimum of the price. Their distribution thus captures the way successive drops can influence each other and construct in this way a persistent process, which is not measured by the distribution of returns. Drawdowns are important measures of risks used by practitioners because they represent their cumulative loss since the last estimation of their their wealth. It is indeed a common psychological trait of people to estimate a loss by comparison with the latest maximum wealth. The distribution of drawdowns for independent price increments x is asymptotically an exponential. By analyzing the major financial indices, the major currencies, gold, the twenty largest U.S. companies in terms of capitalisation as well as nine others chosen randomly, it has been found [3] that approximately 98% of the distributions of drawdowns is well-represented by an exponential

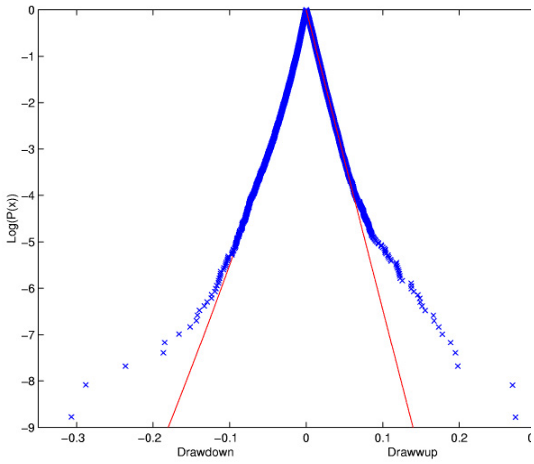

Figure 2: Normalized natural logarithm of the cumulative distribution of drawdowns and of the complementary cumulative distribution of drawups for the Dow Jones Industrial Average index (US stock market). The two continuous lines show the fits of these two distributions with the stretched exponential distribution. Negative values such as −0.20 and −0.10 correspond to drawdowns of amplitude respectively equal to 20% and 10%. Similarly, positive values corresponds to drawups with, for instance, a number 0.2 meaning a drawup of +20%. Reproduced from [1]

or a stretched exponential, while the largest to the few ten largest drawdowns are occurring with a significantly larger rate than predicted by the exponential. The same can be seen in Figure 2. Figure 2 shows the distribution of drawdowns for the returns of the DJIA over last century. Typical market days obey the stretched exponential model well. However, very large drawdowns still deviate significantly. This implies that crash days significantly differ in their statistical properties from the typical market days. Very large drawdowns thus belong to a different class of their own and call for a specific amplification mechanism. The very largest drawdowns are outliers and they can only be explained by invoking the emergence of rare but sudden persistence of successive daily drops, with in addition correlated amplification of the drops. Why such successions of correlated daily moves occur is a very important question.

3 Mechanism

Any deviation from a random walk (typical market days) in the stock market price trajectory has ultimately to be traced back to the behavior of investors. In particular what is interested in is the mechanisms that may lead to positive feedbacks on prices, i.e., to the fact that, conditioned on the observation that the market has recently moved up (respectively down), this makes it more probable to keep it moving up (respectively down), so that a large cumulative move ensues. The positive feedback leads to speculative trends which may dominate over fundamental beliefs and which make the system increasingly susceptible to any exogenous shock, thus eventually precipitating a crash. It has been stressed the importance of positive feedback in a dynamical theory of asset price bubbles that exhibits the appearance of bubbles and their subsequent crashes [4].

There are many mechanisms in the stock market and in the behavior of investors which may lead to positive feedbacks, like "herd" or "crowd" effect, based on imitation processes. There are growing empirical evidences of the existence of herd or "crowd" behavior in speculative markets [5]. Herd behavior is often said to occur when many people take the same action, because some mimic the actions of others. It is the result of the repetitive interaction, which can be easily understood in a network model.

All the traders in the world are organized into a network of family, friends, colleagues, contacts, and so on, which are sources of opinion and they influence each other *locally* through this network [6]. Suppose Yu is the name of an agent. "Neighbors" of agent Yu on this world-wide graph are called the set of people in direct contact with Yu. Other sources of influence also involve newspapers, web sites, TV stations, and so on. Specifically, if Yu is directly connected with k "neighbors" in the worldwide graph of connections, then there are only two forces that influence Yu's opinion: (a) the opinions of these k people together with the influence of the media; and (b) an idiosyncratic signal that he alone receives (or generates). According to the concept of herding and imitation, the *assumption* is that agents tend to *imitate* the opinions of their "neighbors", not contradict them. It is easy to see that force (a) will tend to create order, while force (b) will tend to create disorder. The consequences of the fight between order and disorder are expected to appear some critical phenomena.

It can be put into a more formalized way: each one can be named by an integer $i = 1, \ldots, I$, and $N(i)$ denotes the set of the agents who are directly connected to agent

i according to the world-wide graph of acquaintances. If one trader, Yu, is picked out, $N(\text{Yu})$ is the number of traders in direct contact with him and who can exchange direct information with him and exert a direct influence on him. For simplicity, it is assumed that any investor such as Yu can be in only one of several possible states. In the simplest version, only two possible states are considered: $s_{\text{Yu}} = -1$ or $s_{\text{Yu}} = +1$. They could be interpreted as "buy" and "sell", "bullish" and "bearish", "optimistic" and "pessimistic", and so on. Based only on the information of the actions $s_j(t-1)$ performed yesterday (at time $t-1$) by his $N(\text{Yu})$ "neighbors", Yu maximizes his return by having taken yesterday the decision $s_{\text{Yu}}(t-1)$ given by the sign of the sum of the actions of all his "neighbors". In other words, the optimal decision of Yu, based on the local polling of his "neighbors" who he hopes represents a sufficiently faithful representation of the market mood, is to imitate the majority of his neighbors. This is of course up to some possible deviations when he decides to follow his own idiosyncratic "intuition" rather than being influenced by his "neighbors". Such an idiosyncratic move can be captured in this model by a stochastic component independent of the decisions of the neighbors or of any other agent. Intuitively, the reason why it is in general optimal for Yu to follow the opinion of the majority is simply because prices move in that direction, forced by the law of supply and demand. This apparently innocuous evolution law produces remarkable self-organizing patterns. Consider N traders in a network, whose links represent the communication channels through which the traders exchange information. The graph describes the chain of intermediate acquaintances between any two people in the world. The traders buy or sell one asset at price $p(t)$ which evolves as a function of time assumed to be discrete and measured in units of the time step Δt. In the simplest version of the model, each agent can either buy or sell only one unit of the asset. This is quantified by the buy state $s_i = +1$ or the sell state $s_i = -1$. Each agent can trade at time $t-1$ at the price $p(t-1)$ based on all previous information including that at $t-1$. The asset price variation is taken simply proportional to the aggregate sum $\sum_{i=1}^{N} s_i(t-1)$ of all traders' actions: indeed, if this sum is zero, there are as many buyers as they are sellers and the price does not change since there is a perfect balance between supply and demand. If, on the other hand, the sum is positive, there are more buy orders than sell orders, the price has to increase to balance the supply and the demand, as the asset is too rare to satisfy all the demand. There are many other influences impacting the price change from one day to the other, and this can usually be accounted for in a simple way by adding a stochastic component to the price variation. This term alone would give the usual log-normal random walk process [7] while the balance between supply and demand together with imitation leads to some organization.

At time $t-1$, just when the price $p(t-1)$ has been announced, the trader i defines his strategy $s_i(t-1)$ that he will hold from $t-1$ to t, thus realizing the profit (or loss) equal to the price difference $(p(t) - p(t-1))$ times his position $s_i(t-1)$. To define his optimal strategy $s_i(t-1)$, the trader should calculate his expected profit P_E, given the past information and his position, and then choose $s_i(t-1)$ such that P_E is maximum. Since the price moves with the general opinion $\sum_{i=1}^{N} s_i(t-1)$, the best strategy is to buy if it is positive and sell if it is negative. The difficulty is that a given trader cannot poll the positions s_j that will take all other traders which will determine the price drift according to the balance between supply and demand. The next best thing that trader i can do is to poll his $N(i)$ "neighbors" and construct his prediction for the price drift from this information. The trader needs an additional information, namely the a priori probability P_+ and P_-

for each trader to buy or sell. The probabilities P_+ and P_- are the only information that he can use for all the traders that he does not poll directly. From this, he can form his expectation of the price change. The simplest case corresponds to a market *without drift* where $P_+ = P_- = 1/2$.

Based on the previously stated rule that the price variation is proportional to the sum of actions of traders, the best guess of trader i is that the future price change will be proportional to the sum of the actions of his neighbors that he has been able to poll, hoping that this provides a sufficiently reliable sample of the total population. It is then clear that the strategy that maximizes his expected profit is such that his position is of the sign given by the sum of the actions of all his "neighbors". This is exactly the meaning of expression (1)

$$s_i(t-1) = \text{sign}\left(K \sum_{j \in N_i} s_j + \varepsilon_i \right) \tag{1}$$

such that this position $s_i(t-1)$ gives his the maximum payoff based on his best prediction of the price variation $p(t) - p(t-1)$ from yesterday to today. The function sign(x) is defined by being equal to $+1$ (to -1) for positive (negative) argument x, K is a positive constant of proportionality between the price change and the aggregate buy-sell orders. It is inversely proportional to the "market depth": the larger the market, the smaller is the relative impact of a given unbalance between buy and sell orders, hence the smaller is the price change. ε_i is a noise and $N(i)$ is the number of neighbors with whom trader i interacts significantly. In simple terms, this law (1) states that the best investment decision for a given trader is to take that of the majority of his neighbors, up to some uncertainly (noise) capturing the possibility that the majority of his neighbors might give an incorrect prediction of the behavior of the total market.

The imitative behavior discussed and captured by the expression (1) belongs to a very general class of stochastic dynamical models developed to describe interacting elements, particles, agents in a large variety of contexts, in particular in physics and biology [8][9]. The tendency or force towards imitation is governed by the coupling strength K; the tendency towards idiosyncratic (or noisy) behavior is governed by the amplitude ε_i of the noise term. Thus the value of K relative to ε_i determines the outcome of the battle between order and disorder, and eventually the structure of the market prices.

The above model can be mapped to the simplest possible network of a two-dimensional grid in the Euclidean plane. Each agent has four nearest neighbors: one to the North, one to the South, the East and the West. The tendency K towards imitation is balanced by the tendency ε_i towards idiosyncratic behavior. In the context of the alignment of atomic spins to create magnetisation (magnets), this model is identical to the two-dimensional Ising model which has been solved explicitly by Onsager.

In the Ising model, there exists a critical point K_c that determines the properties of the system, below which disorder reigns: the sensitivity to a small global influence is small, the clusters of agents who are in agreement remain of small size, and imitation only propagates between close neighbors and approaching which order starts to appear: the system becomes extremely sensitive to a small global perturbation, agents who agree with each other form large clusters, and imitation propagates over long distances. Formally, in this case the susceptibility χ of the system goes to infinity. The large susceptibility means that the system is unstable: a small external perturbation may lead to a large collective reaction of the traders who may revise drastically their decision, which may abruptly produce a sudden unbalance between supply and demand, thus triggering a crash.

Indeed, the stock market constitutes an ensemble of interacting investors who differ in size by many orders of magnitudes ranging from individuals to gigantic professional investors, such as pension funds. Furthermore, structures at even higher levels, such as currency influence spheres (US$, DM, YEN ...), exist and with the current globalization and de-regulation of the market one may argue that structures on the largest possible scale, *i.e.*, the world economy, are beginning to form. This observation and the network of connections between traders show that the two-dimensional lattice representation used is too naive. A better representation of the structure of the financial markets is that of hierarchical systems with "traders" on all levels of the market. Of course, this does not imply that any strict hierarchical structure of the stock market exists, but there are numerous examples of qualitatively hierarchical structures in society. In fact, one may say that horizontal organizations of individuals are rather rare. This means that the plane network used in the previous discussion may very well represent a gross over-simplification.

Even though the predictions of these models are quite detailed, they are very robust to model misspecification. It is claimed that models that combine the following features would display the same characteristics, in particular apparent coordinate buying and selling periods, leading eventually to several financial crashes. These features are: (1) a system of traders who are influenced by their "neighbors"; (2) local imitation propagating spontaneously into global cooperation; (3) global cooperation among noise traders causing collective behavior; (4) Prices related to the properties of this system; (5) system parameters evolving slowly through time. As to be shown in the following sections, a crash is most likely when the locally imitative system goes through a *critical* point.

4 Modeling

A kind of microscopic explanation is given previously to the market crashes. Now it is time for a phenomenological treatment. A crash is not certain but can be characterized by its hazard rate $h(t)$, i.e., the probability per unit time that the crash will happen in the next instant if it has not happened yet, which has influence on the trend of prices. The crash hazard rate $h(t)$ embodies subtle uncertainties of the market, like the time when other traders think that a crash is coming. As far as asset prices are concerned, a crash happens when order wins (due to imitation everybody has the same opinion: selling), and normal times are when disorder wins (buyers and sellers disagree with each other and roughly balance each other out).

In the spirit of "mean field" theory of collective systems, the simplest way to describe an imitation process is to assume that the hazard rate $h(t)$ evolves according to the following equation:

$$\frac{dh}{dt} = C\ h^{\delta}\ , \qquad \text{with } \delta > 1\ , \tag{2}$$

where C is a positive constant. So the hazard rate enhances itself. Mean field theory amounts to embody the diversity of trader actions by a single effective representative behavior determined from an average interaction between the traders. In this sense, $h(t)$ is the collective result of the interactions between traders. The term h^{δ} in the r.h.s. of (2) accounts for the fact that the hazard rate will increase or decrease due to the presence of *interactions* between the traders. The exponent $\delta > 1$ quantifies the effective number equal to $\delta - 1$ of interactions felt by a typical trader. Indeed, integrating (2), it is given

$$h(t) = \frac{B}{(t_c - t)^{\alpha}} \ , \qquad \text{with } \alpha \equiv \frac{1}{\delta - 1} \ . \tag{3}$$

The critical time t_c is determined by the initial conditions at some origin of time. The exponent α must lie between zero and one for an economic reason[]. This condition translates into $2 < \delta < +\infty$: a typical trader must be connected to more than one other trader. t_c is not the time of the crash, it is the end of the bubble. It is the most probable time of the crash because the hazard rate is largest at that time. Due to its probabilistic nature, the crash can occur at any other time, with a likelihood changing with time following the crash hazard rate. There exists a finite probability

$$1 - \int_{t_0}^{t_c} h(t)dt > 0 \tag{4}$$

of "landing" smoothly, i.e., of attaining the end of the bubble without crash. This residual probability is crucial for the coherence of the model, because otherwise agents would anticipate the crash and would exit from the market.

Furthermore, it is predicted that the critical exponent α can be a complex number[]! The first order expansion of the general solution for the hazard rate is then

$$h(t) \approx B_0(t_c - t)^{-\alpha} + B_1(t_c - t)^{-\alpha} \cos[\omega \log(t_c - t) - \psi]. \tag{5}$$

where ψ and ω are some constants. Once again, the crash hazard rate explodes near the critical date. In addition, it now displays log-periodic oscillations.

To relate the hazard rate and the price trend, a linear response approximation is taken before crashes:

$$dp = \kappa[p(t) - p_1]h(t)\, dt, \tag{6}$$

where p_1 is a reference value. The evolution of the price before the crash and before the critical date is given by:

$$p(t) \approx p_c - \frac{\kappa}{z} \left\{B_0(t_c - t)^z + B_1(t_c - t)^z \cos[\omega \log(t_c - t) - \phi]\right\} \tag{7}$$

where $z = 1 - \alpha \in (0, 1)$, p_c is the price at the critical time (conditioned on no crash having been triggered). The key feature is that oscillations appear in the price of the asset before the critical date. This means that the local maxima of the function are separated by time intervals that tend to zero at the critical date, and do so in geometric progression, i.e., the ratio of consecutive time intervals between maxima is a constant

$$\lambda \equiv e^{\frac{2\pi}{\omega}} \ . \tag{8}$$

This is very useful from an empirical point of view because such oscillations are much more strikingly visible in actual data than a simple power law: a fit can "lock-in" on the oscillations which contain information about the critical date t_c.

The logic of the above risk-driven model is that the stock market price is driven by the risk of a crash, quantified by its hazard rate. In turn, imitation and herding forces drive the crash hazard rate. When the imitation strength becomes close to a critical value, the crash hazard rate diverges with a characteristic power law behavior. This leads to a specific power law acceleration of the market price, with log-periodic oscillations. This model captures some main features of stock market crashes, and it agrees quite well with the data of 1987 crash [1].

However, the phenomenological model presented here does not probe the spatial correlation at all. It simply assumes the existence of the collective behavior, and can not describe the formation of this kind of collective behavior. And the singularity of the result of the model is, in some sense, put by hand. It is the same that can be expected to almost all positive feedback system. Though it is a long time scale model, the initial condition applied to the solution is ambiguous. It only accounts the time before the crash, but incapable of describing the period after the crash, not mention how to relate the next crash with the precedent one.

5 Summary

The normal stock market days are believed to be different from the crash days. The drawdowns prove to be a better way than the traditional daily return to dismantle the emergent long temporal correlation of the stock market. This places the basis of the hypothesis that large stock market crashes are analogous to critical points, familiar in statistical physics. The underlying mechanism is conjectured as the cooperative behavior of traders imitating each other. Based on the microscopic argument, with great simplification, a phenomenological mathematical model is constructed here. A general result of the model is the existence of log-periodic structures decorating the time evolution of the system. The main point is that the market anticipates the crash in a subtle self-organized and cooperative fashion, hence releasing precursory "fingerprints" observable in the stock market prices. In other words, this implies that market prices contain information on impending crashes.

There are a lot of models proposed for crashes which have pondered the possible mechanisms to explain the collapse of the price at very short time scales [10]. In contrast, here, the underlying cause of the crash is proposed to be searched years before it in the progressive accelerating ascent of the market price, reflecting an increasing build-up of the market cooperatively. From that point of view, the specific manner by which prices collapsed is not of real importance since, according to the concept of the critical point, any small disturbance or process may have triggered the instability. The intrinsic divergence of the sensitivity and the growing instability of the market close to a critical point might explain why attempts to unravel the local origin of the crash have been so diverse. Essentially all would work once the system is ripe. So what is really of importance is the endogeneous origin of crashes, and exogeneous shocks only serve as triggering factors. It is proposed [1] that the origin of the crash is much more subtle and is constructed progressively by the market as a whole. In this sense, this could be termed a systemic instability.

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