Turbulence in Fluids *

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Abstract

The basic dynamical equations for incompressible fluid flow (the Navier-Stokes equations) are known for the last one and a half centuries. Yet a detailed understanding of flow at high velocities (Reynolds numbers) remains elusive. The basic assumptions of the "classical" theory due to Kolmogorov is not beyond question. Here I will introduce basic questions related to the problem of turbulence, discuss on Kolmogorov's theory of turbulence and report briefly and selectively on the work that has been done after that.

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1 Introduction: What is turbulence?

There is an apocryphal anecdote that has been related of more than one scientist, but should probably be attributed to Heisenberg. When asked what he would ask God if given an opportunity, he supposedly said that he would ask two questions "Why relativity? And why turbulence? I really believe he will have an answer for the first." Indeed in the eyes of many the problem of fluid turbulence remains the last great unsolved problem in classical physics. The complexity of the problem can be gauged from the fact that even mere characterisation is non trivial. But before proceeding let us have a look at what is universally believed to be the 'microscopic' equations for the phenomenon.

The dynamical flow of most liquids and also some gases to a large degree of accuracy seem to obey the Navier-Stokes equations. Here I will consider only these so called Newtonian fluids were it is assumed that the stresses are proportional strains, and that the strains are given by the velocity gradients (as opposed to the displacement gradients as in solids). Also one believes that it is reasonable to treat the fluid bulk as a continuum, since the smallest length scales of flow are much larger than the intermolecular distances. At least for flows with velocities less than the speed of sound in the medium, this approximation has no reason to run into trouble. Thus the theory is analogous to the theory of elasticity for solids. We make the further simplifying assumptions that the fluid is incompressible, *i.e.* density, $\rho =$ constant, and that the temperature gradient across the fluid body is negligible[1]. With the above assumptions the Navier-Stokes equations take the following form,

$$\frac{\partial \tilde{u}_i}{\partial t} = -\tilde{u_k}\frac{\partial \tilde{u}_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_k \partial x_k} \tag{1}$$

It is supplemented by the continuity equation,

$$\frac{\partial \tilde{u}_k}{\partial x_k} = 0 \tag{2}$$

Here $\tilde{u}_i(\mathbf{x}, t)$ and $\tilde{p}(\mathbf{x}, t)$ represent the i th Cartesian component of the velocity field and the pressure at position $\mathbf{x}$ and time t respectively. Also ν is the so called kinematic viscosity. The NS equations are merely a restatement

[1] These assumptions are reasonable when the typical velocities in the fluid are much less compared to the velocity of sound in the fluid

of Newton's second law for fluid systems, with additional internal friction which is assumed to obey Newton's hypothesis of viscosity. Often one derives these equations semi-rigorously from collisional gas dynamics, but one must remember that that represents a very special case of dilute systems; the realm of validity of the NS equations is much wider than those of its 'derivations'. It is interesting to note that one can derive these equations from very general symmetry arguments and the assumption that the only important quantities in a fluid are the velocity field $\mathbf{v}$ and the pressure head p. One more important point to note is that if one agrees to measure velocity and length in the units of some typical length L and velocity V of the system then the flow, at least sufficiently far form the boundary, will depend only on a dimensionless parameter

$$R = \frac{VL}{\nu} \tag{3}$$

R is called the Reynolds number for the system. Now when R is small, the typical velocities for the system are small, which implies that the non linear term in the equations becomes small compared to the other terms. For such cases it is sensible to characterise the flows in terms of stable, stationary slowly varying flows. Such flows are called laminar flows. The auto correlation function of the velocity over time (when suitably scaled) is of the order unity. Such flows are rarely seen unless under carefully constructed conditions. Now if one starts increasing the Reynolds number, then the smooth stationary flow starts breaking up. The velocity at each point starts showing time dependence. On further increasing the Reynolds number, one goes through a regime when the velocity field shows bursts of very rapid time variation seperated by periods of very little or no time dependence. This phenomenon is called *intermittency*. Finally at extremely large Reynolds number the velocity field (apparently) loses all correlation over any finite interval of time. It becomes practically random. This phase is the the so called *fully developed turbulence*. A set of cartoons for the above mentioned changes with increasing Reynolds number is shown in Fig.1. A word of caution: the picture sketched above is schematic and emperical and the actual evolution of a system with increasing Reynolds number is heavily system dependent. For example, similar experiments when performed in a liquid placed in the region between rotating concentric cylinders leads to the so called Taylor-Couette flow. The sequence of events in that case are analogous but the spatial flow patterns are dissimilar to the one shown in the figure.

2 Major questions related to turbulence

There are a few broad questions which need to be addressed. Firstly, what is the path taken by the system to turbulence? How much of it is global? Or, what is more appropriate, are there any global features? The uniqueness or smoothness of solutions for the NS equations for arbitary time has not been established. As such it is unclear whether the transtion to turbulence is due to the change of stability of one branch of the solutions to another, or is it merely due to the evolution of the same branch of the solutions with increasing Reynolds number, *i.e.* is turbulence a *new state* of the fluid system which is absent below a certain critical Reynolds number or is it a natural evolution of the same *state*. In case the former is true one should like to know how this *critical* Reynolds number depends on the flow geometry etc. Indications are that if the former is indeed true (*i.e.* if there indeed exists a critical Reynolds number) then it is critically dependent on the characteristics of the system. It has been shown experimentally that under sufficiently careful conditions the onset of *fully developed turbulence* with increasing Reynolds number can be delayed to a large extent.

The last observation brings us to the next important category of questions. Is turbulence indeed a characteristic of the NS equations or is it merely the effect of the randomnessof the boundary conditions or other other external conditions[2]. Attempts so far to *derive* turbulence from the NS equations in three dimensions, have not been remotely successful so far. However other approaches have been tried. One approach is to assume that the description of turbulence should be done at a level higher than the description of the system offered in the NS equations. An example will make the situation clearer. The NS equations describe a wide class of fluids. As mentioned earlier, one can *derive* them form molecular dynamics. However that is not the whole story. The NS can also be *derived* using a variety of other microscopic dynamics, viz. cellular automata etc. In practice the NS equations are just a statement about the symmetries of the (fluid) system, which is (in the contin-

[2]It is well known that turbulence cannot be sustained in an isolated system, since the fluid system is inherently dissipative. One needs to provide an external driving force to sustain turbulence.The randomness in this force, amplified by the nonlinear couplings of the NS equations may be the cause of turbulence itself.

uum limit) quite independent of the microscopic dynamics of the system up to a finite number of experimantally determined parameters. Indeed the NS equations are in a sense more *global* then the specific microscopic dynamics. Now using an analogous string of arguments one can say that turbulence is a global phenomenon of which the NS equation is just one *microscopic* description. In principle one can construct other such *microscopic* descriptions which will give turbulence. The global features of turbulence, if any, should not depend on the details of the microscopics. Several such turbulence models have been attempted. Here as an example, I will consider a simple one dimensional model due to Bass [15].

Let us consider the Burgers equation given by[3],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{4}$$

This equation can be transformed into a linear (heat transfer) equation by making the following transformation

$$u = -\frac{1}{2}\frac{1}{Z}\frac{\partial Z}{\partial X} \tag{5}$$

In which case Z is given by

$$\frac{\partial Z}{\partial t} = -\frac{1}{2}\frac{\partial^2 Z}{\partial x^2} \tag{6}$$

Obviously, the solutions to this equation can be constructed from plane waves and with proper choice of coefficients one can construct u from 5 which will be smooth, bounded and with finite autocorrelation over arbitarily long time intervals. Let us call these solutions *normal*. For certain initial and boundary conditions the system will admit these solutions. However now consider a function $Y(t)$ constructed in the following manner. For $t < 0$, $Y(t)$ is identically zero. When $t > 0$ $Y(t)$ can be either 1 or -1, and it changes from one value to the other arbitarily [4]. In that case it is easy to see that the

[3]This equation has no term analogous to the pressure. It was modeled to give insight about the turbulence in an one dinemsional fluid "which is infinitely compressible, and which is endowed with a cooling mechanism such that the temperature term always remains zero[15]."

[4]It is interesting to note that Bass is reluctant to attribute this arbitariness to statistical randomness. But as pointed out by Burgers in the discussion in the end, it is

autocorrelation function for $Y(t)$ defined as

$$\begin{aligned}\gamma(h) &\equiv \lim_{T\to\infty}\frac{1}{T}\int_0^\infty Y(t)Y(t+h)\mathrm{d}t \\ &= \begin{array}{ll} 1-h & 0\le h<1 \\ 0 & h\ge 1 \end{array}\end{aligned}$$

One can show that Burgers equation has continuous and derivable solutions of the form

$$u(x,t)=\int_0^\infty Y(t+h)\phi(h,x)\mathrm{d}h \tag{7}$$

where ϕ is a square integrable function of its first argument. Let us call these kinds of solutions *anomalous*. Evidently the autocorrelation function for u vanishes after a finite time. Now one can in principle construct initial and boundary conditions for which the system will admit these anomalous solutions. If one identifies this with a one dimensional fluid then one can see that it indeed does lead to *turbulence*, if defined by the autocorrelation functions.

This construction is a very simple example of a turbulence model, but it has many of the generic features of the latter. Often there is implicit or explicit reduction of dimensions. While this approach may manage to capture the some of the essential physics, it should be remembered that turbulence is the response of a system consisting of many degrees of freedom to very strong self coupling and possibly coupling with external sources. Also a turbulent flow is characterised by very rapid evolution of eddies of different sizes. Now this is not possible in any dimension less than three. Moreover it is likely that with mere simplification a loss of essential physics may be unavoidable.

The last important category of problems connected to turbulence relates to the characterisation of *fully developed turbulence*. In the next section I will discuss certain aspects of it.

3 Fully developed turbulence

The turbulent regime is emperically characterised by rapid time evolution over a large range of length scales. Hence an initial value problem approach

difficult to visulise a situation other than ensemble averaging which will give rise to such an arbitariness.

seems hopeless. On the other hand a stochastic treatment is likely to be more profitable. To that end separate the mean and the fluctuating parts of the fluid variables.

$$\tilde{u}_i = U_i + u_i \tag{8}$$

$$\tilde{p} = P + p \tag{9}$$

The upper case variable on the right hand sides of both equations represent the averaged quantities and the lower case variables represent the corresponding fluctuations. The averages indicated here are ensemble averages, but from the ergodic hypothesis they will be equal to the time averages over a long enough time[5]. By definition the averages of u_i and p are identically zero.

One can in principle write down equations for the averaged quantities by taking an average over the Navier-Stokes equations and also keeping in mind that the continuity condition is seperately obeyed by the mean and the fluctuations of the velocity. However it is easy to see that this equation will contain terms like $\langle u_i u_j \rangle$ (which is in general non zero)where $\langle \ldots \rangle$ imply ensemble average.

Again it is possible to write equations for $\langle u_i u_j \rangle$, and for that matter $\langle u_i u_j u_k \rangle$ and so on by taking the appropriate moments of the NS equation and averaging. But a moments look will convince one that the equation for each such averaged moment involves a higher order moment. Thus this system of equations (infinite in number) runs into serious closure problems, to which no satisfactory solution has yet been found except certain ad hoc truncations of moments higher than a certain order to force closure.

4 The Kolmogorov hypothesis

Evidently the NS equations despite of being complete by themselves do not allow for much predictability by the sheer magnitude of its complexity. One realises that some new input about the nature of turbulence is required to make any progress. Arguably the first such attempt was made by Kolmogorov[1, 3], and his hypothesis has ever since served as the benchmark for all other attempts to follow.

[5]The definition of an ensemble is far form simple in this case. Neither is the invocation of the ergodic hypothesis free from debate. However I will be content with just stating this problem.

To motivate the Kolmogorov hypothesis let us define a new quantity

$$R_{ij} = \langle v_i(\mathbf{x}, t), v_j(\mathbf{x} + \mathbf{r}, t) \rangle \tag{10}$$

i.e. the equal time two point velocity correlation function. (Henceforth we will supress the dependence on time.) The first assumption that we make is that the above defined correlation function is isotropic and homogenous. This should be a reasonably good approximation in the region of the fluid which is far from any material boundary. From now on we will concentrate on this idealised region only. It is easy to see that with those approximations and the incompressibility condition,

$$R_{ij} \equiv R_{ij}(r) = A(r) r_i r_j + B(r) \delta_{ij} \tag{11}$$

Now let us take the Fourier transform of R_{ij}

$$D_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int R_{ij} e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3\mathbf{r} \tag{12}$$

This means

$$D_{ij} \equiv D_{ij}(k) = E(k) k_i k_j + F(k) \delta_{ij} \tag{13}$$

From the incompressibility condition

$$k_i D_{ij} = k_j D_{ij} = 0 \tag{14}$$

we get,

$$F(k) = E(k) k^2 \tag{15}$$

Hence with these simplifying assumptions were are left with the job of determining only one scalar function. The physical significance of this function can be seen from the following relation

$$\begin{aligned} \frac{1}{2}\langle \mathbf{v}^2 \rangle &= \frac{1}{2} R_{ii}(0) \\ &= \int D_{ij}(k) \mathrm{d}^3\mathbf{k} \\ &= \int_0^\infty E(k) \mathrm{d}k \end{aligned} \tag{16}$$

This means one should regard $E(k)$ as the energy spectrum of isotropic, homogenous turbulence. Due to the presence of the dissipative viscosity term, turbulence in an isolated system is not sustained. There has to be an external energy source to keep it alive. Assume that this source feeds energy to the system at a length scale of L and a velocity scale of V. Thus the largest scale Reynolds number of the system is

$$R = \frac{LV}{\nu} \tag{17}$$

Let the source supply energy at a rate of ϵ. Kolmogorov put forward the following hypothesis. He postulated that this energy is transmitted to this highest length scale to successively lower length scales, and dissipated only at the lowest allowed length (l_d) and velocity (v_d)scale in the system, determined by the molecular dynamics of the system.This is identified as the scale where the Reynolds number becomes of order unity.

$$l_d v_d \sim \nu \tag{18}$$

Kolmogorov also postulated that one should be able to write ϵ only in terms of the typical length(l) and velocity(v) scale associated with any intermediate eddy. From dimensional grounds, the only way to do this is

$$\epsilon \sim \frac{v^3}{l} \tag{19}$$

Since this is true for the highest and the lowest length scales,

$$\frac{L}{l_d} \sim R^{3/4} \tag{20}$$

$$\frac{V}{v_d} \sim R^{1/4} \tag{21}$$

Now considering the spectrum we can easily see that a higher and a lower cutoff wavenumber will be introduced by the lowest and the highest length-scale in the problem. In the intermediate region, sufficiently far away from either of the cutoffs (this regime is called the inertial range), we can again write from dimensional arguments,

$$E(k) = C\epsilon^{2/3} k^{-5/3} \tag{22}$$

Where C is a dimensional number with only possible dependence on the largest scale Reynolds number of the system (R). Again Kolmogorov predicted that for sufficiently large Reynolds numbers, C becomes independent of R. In practice this is a statement postulating the existence of the limit $C(R)$ for $R \to \infty$. With this qualification of existence of limit, the last equation is the famous Kolmogorov five-thirds scaling law for the inertial range Fig.2.

5 Further considerations

Looking at the apparent crudeness of the approximations which goes into the *derivation* of the scaling law, its corroboration by experiments have been good enough, if one agrees to view only the rought general trends. The goodness of its exact fit with the expertimental data points is is not very clear, because the data contains considerable scatter. Sreevasan has done a detailed analysis of the available data for grid generated turbulence[2]. I will reproduce his results in Fig.3. The results in this graph show that the Kolmogorov constant is not universal. However a systemetic set of experiments for a single flow geometry are yet to be done.[7] The last assumption in Kolmogorov's hypothesis has been the topic of much debate. In fact, as first pointed out by Landau, it is highly unlikely that averaging the rate of energy transfer (*cascade*) across nonuniversal large scales will not give rise to any spatial structure. There have been a number of attempts to come up with a suitable modified hypothesis or at least with a better and more complete understanding of the underlying phenomena. Kolmogorov himself had comeup with two further refinements to the so called K41 hypothesis. Here as a prototype example I will reproduce the analysis of Goldenfeld and Barenblatt [7, 8, 9]. Firstly,it should be noted that the five-thirds scaling law also presupposes an implicit asymptotic constant nonzero limit of C as $kL \to \infty$, because in principle C can as well be a function of kL. However it can be argued that C can depend on R only through $\ln R$ and on k only through kL. The latter is easy to see, the former can be understood intuitively in the following manner. The Reynolds number for a flow is not a perfectly well defined quantity. One can in principle define more than one Reynolds number for the same flow differing at the most by factors of order unity, or stated otherwise one can redefine the Reynolds number upto a factor of order unity for the same physical flow. However this should not change the flow variables, in particular, the correlation function D_{ij}, at least not upto the the leading order for $R \to \infty$. A little thought will convince one that in that case the leading dependence on R can only be through $\ln R$. In fact this result can also be interpreted as a statement of weak dependence of the spatial flow properties on a less than well defined parameter. A fairly general functional form for C which the authors consider is

$$C(R, kL) = A(\ln R)(kL)^{\alpha(\ln R)} \tag{23}$$

Now A and α can be conveniently expanded in the small parameter $\beta = \frac{1}{R}$. If it is assumed that the leading constant terms are non zero then one gets back the Kolmogorov hypothesis. However when the leading term is assumed to be linear in β then as expected one obtains a family of curves in the D_{ij} vs kL space, parametrised by R. The envelope of this family is universal, although the individual members are not.

The above example is instructive for the following reason. The Kolmogorov hypothesis for turbulence is analogous to the Landau theory critical phenomena, in the sense that both are mean field theories. It is known that the Landau theory gives quantitatively incorrect results for most systems undergoing phase transitions. However that is not unexpected. The Landau theory does not take into account the fluctuations which become important near phase transitions. But the importance of Landau theory lies not in precise quantitative predictions, but in the fact that it allowss one to identify the gross macroscopic quantities in the system from (mostly) symmetry considerations only. It provides a completely different *top-down* approach as opposed to the standard *bottom-up* approach. Similar considerations should possibly apply to the Kolmogorov hypothesis of 1941.

6 Turbululence and critical phenomena

The discussion at the end of the last section brings one to the interesting analogies that can be drawn between turbulence and critical phenomena. Whether this analogy has any physical meaning or whether it is merely formal, is unclear. As briefly discussed previously, it is not obvious if turbulence is indeed a new emergent state in statistical equilibrium, in which case one should talk about the (phase) transition from laminar flow to turbulent flow, or is it merely a smooth evolution or crossover from laminar flow.

However, regardless of the underlying physics, the application of the machinery developed for studying critical phenomena towards analysing the problem of turbulence has enough computational merit. In particular,the future for the application of renormalisation group techniques seem promising. However so far efforts towards the above generalisation have proved to be non trivial and is running into their share of complications.

The discussion of RG in turbulence to any depth is beyond the scope of this report; I will simply state some of the formal analogies that exists between critical phenomena and turbulent flow[6]. To this end, the first important thing is to notice that spatial separation ($\mathbf{r}$) in systems undergoing critical phenomena is analogous to wavenumber ($\mathbf{k}$) in the case of turbulence. Most of the other analogies follow from this observation. The length scale of energy insertion (L), the length scale of energy dissipation(l_d) and the velocity correlation function (R_{ij}) in turbulence are respectively analogous to inverse lattice spacing, inverse correlation length and the Fourier transform of the one particle Greens function in systems undergoing critical phenomena[4]. Other identifications which may not be immediately evident can be made as well. The viscosity (ν), the so called *intermittency* exponent (α) and the mean rate of dissipation of energy (ϵ) are similarly analogous to temperature[7], correlation exponent and stiffness[8] respectively[10].

7 Conclusions

In this report I have given a simple, perhaps simplistic, account of the problem of turbulence. Almost no technical details have been given due to lack of space and necessity (and in most cases due to the rather limited knowledge of the author). In some cases this may have resulted in the loss of clarity, as I have, more often than not, merely mentioned pieces of information without discussing them. The problem of turbulence has entered the new millenium

[6]In such calculations one thinks of the fluid system as undergoing two *phase transitions* when the Reynolds number is increased[13] . The first being a transition from laminar flow to turbulent flow and the second from turbulent flow to *fully developed turbulence.* The latter defined as one for which a statistical description is meaningful.

[7]Rather to the temperature difference from its critical value $i.e. T - T_c$. However, I think it may be more appropriate if one draws an analogy between the Reynolds number (R) and the scaled temperature difference $\frac{T-T_c}{T_c}$, since these are the only important dimensionless parameters in the respective systems

[8]only for lattice spin-systems

as probably the only standing unsolved problem in classical physics. Immense amount of literature in physics, mathematics as well as engineering has been devoted to its analysis. I have not even attempted to do justice to the impossible task of mentioning all of them. However, I hope, I have atleast been able to present the major questions that need to be asked (in my view), when one addresses the problem of turbulence in fluids from the perspective of a physicist.

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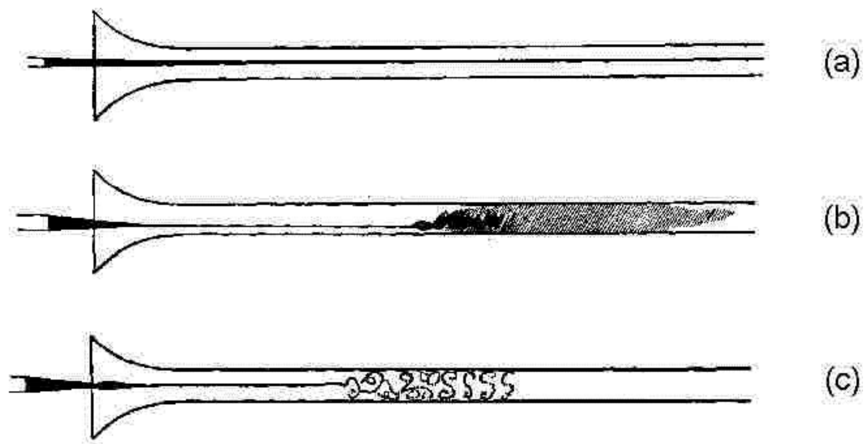

Figure 1: Fluid flow in a cylinder. Schematic of the streamlines with increasing Reynolds number.

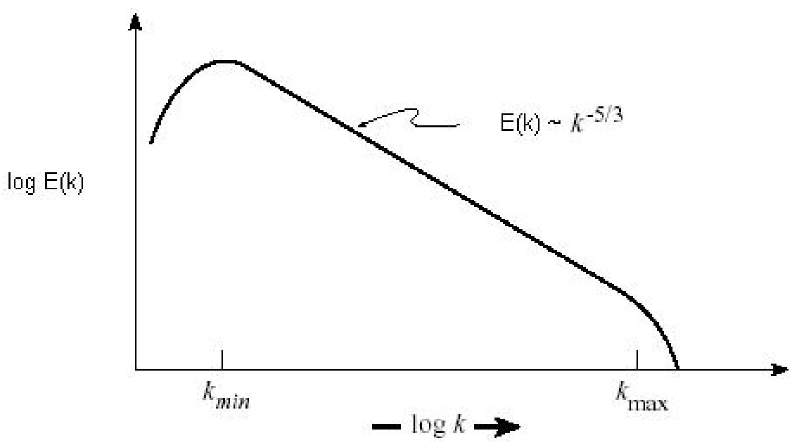

Figure 2: The scaling for Kolmogorov hypothesis of 1941

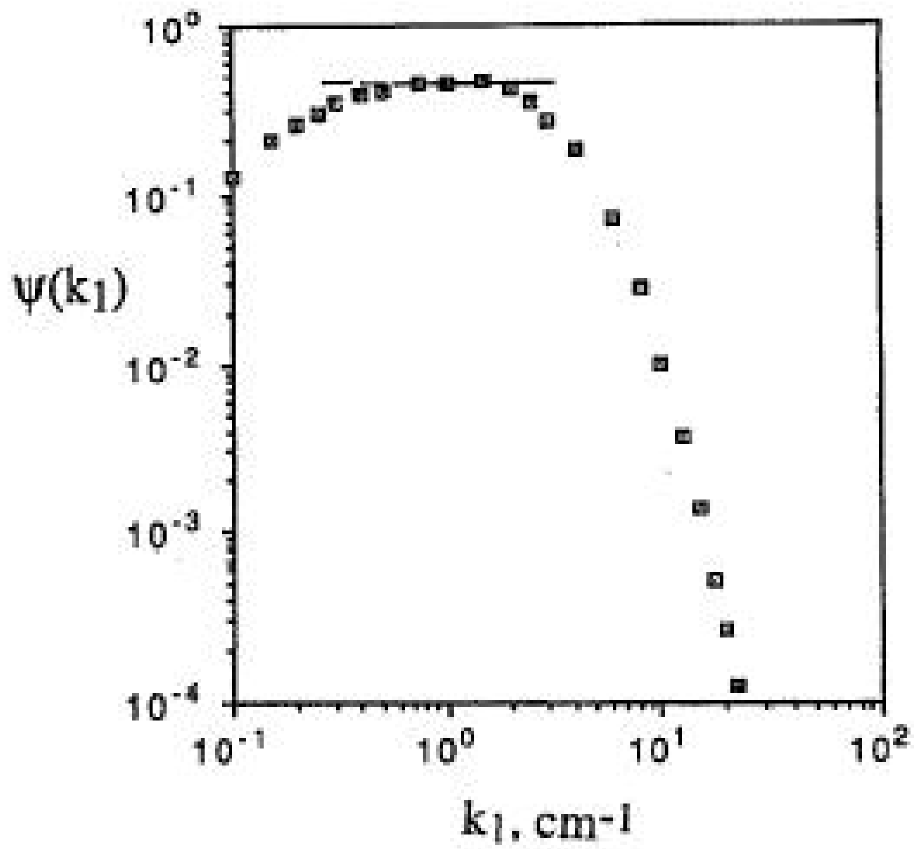

Figure 3: Compensated one dimensional spectral density plotted against wavenumber for one grid-generated turbulence. The three dimensional spectral density is given by $E(k) = k^2 \frac{\partial \psi}{\partial k^2} - k \frac{\partial \psi}{\partial k}$. The relatively small flat region near at low Reynolds numbers yields the Kolmogorov constant. [2]