Bose-Einstein Condensation in Trapped Gases: A Review of Experiments and Theories

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Abstract

Bose-Einstein condensation (BEC) is a phase transition where a *macroscopic* number of particles all go into the *same* quantum state. In recent years, the field of BEC, especially BEC in atomic gases, has grown rapidly thanks to both new experimental techniques and theoretical advances. It has attracted researchers from the communities of atomic physics, statistical physics, quantum optics and condensed matter physics. In this paper, basic theoretical ideas and current experimental progresses about BEC in the alkali gases are reviewed. We will also present the possible directions for future researches in this fast-growing area.

1 Introduction — an Intuitive View of BEC

The history of Bose-Einstein condensation (BEC) dates back to the early stage of the development of quantum theory of matter. At the beginning of the twentieth century, Plank solved the problem of thermal electromagnetic radiation by introducing the idea that the radiators emit the energy in *discrete* energy quanta. This marks the birth of quantum theory.

In 1905 Albert Einstein made a further conclusion that the electromagnetic field itself is quantised and is created and converted in bursts of energy. These bursts are light quanta, which are called photons later. In 1924 Bose sent a paper to Einstein, in which he showed that the Plank distribution law could be derived entirely from a *statistical* argument of photons without resort to the results from classical electrodynamics. Einstein soon realized the importance of this new idea and immediately started to work on this problem himself. In 1924 and 1925 he published two papers in which he developed the full picture of quantum theory of bosonic particles. The concept of Bose-Einstein statistics was born. Today we know that all particles with integer spins show Bose-Einstein statistics, hence called bosons, while those with half-integer spins show Fermi-Dirac statistics, hence called fermions.

The experimental systems we will look at in this paper are collections of neutral alkali atoms. Let us treat the individual atom as a single indivisible entity at the moment. In total there are $Z + A$ fermions in each atom, A nucleons and Z electrons. Since Z is already odd for the alkali elements, a gas of odd-A isotopes will obey Bose-Einstein statistics, while an even-A system will obey Fermi-Dirac statistics. We are mainly interested in the former case from now on.

Einstein noticed that if the number of particles is conserved, even a system composed of *non-interacting* particles will undergo a phase transition at low temperatures in which a *macroscopic* number of particles will condense into the *same* quantum state. This transition is called Bose-Einstein condensation (BEC), which is the main subject of this paper.

For concreteness, let us consider a dilute atomic gas. One of the necessary conditions to have a BEC in this system is extremely low temperatures because at high temperatures, e.g., room temperatures, a dilute gas behaves classically. When the temperature is high, the thermal de Broglie wavelength $\lambda_{dB} = h/\sqrt{2\pi m k_B T}$ is small compared to the spacing between atoms, and we can describe the motions of atoms by their classical trajectories. The thermal de Broglie wavelength λ_{dB} measures the position uncertainty due to thermal momentum distribution of an atomic gas at temperature T. It increases with lowering temperature T. The quantum statistics begins to show effect when λ_{dB} becomes comparable with the interatomic spacing. The atomic wave packets will then overlap and classical mechanics based on describing particles by their trajectories will fail. Depending on which quantum statistics the atoms obey, i.e., whether they are fermions or bosons, the gas will show distinct features.

If the atoms are fermions, gradual cooling will bring the gas closer to a state called "Fermi Sea", in which every lowest possible energy state is occupied by exact one atom. The gas resists being compressed into a smaller volume, a manifestation of the Fermi degeneracy pressure. This pressure is also responsible for keeping white dwarfs and neutron stars from collapsing into black holes. Notice the onset of the Fermi degeneracy is *gradual* as the temperature is lowered. At even lower temperatures, if the effective interation between the atoms is attractive, two fermions can bind into a pair behaving like a boson. This phenomenon is responsible for classical superconductors and superfluid ^{3}He. Some research groups are trying to produce a paired superfluid in ultracold fermionic atomic vapor. We will come back to this later.

On the other hand, if the atoms are bosons, a cloud of atoms will occupy the same quantum state, i.e., a condensate will appear. Notice the condensate appers at a *precise* temperature, which is the reason why BEC is called a phase transition.

The striking difference between fermions and bosons is shown in Fig. 1. The sizes of the atom clouds are reduced as the temperature drops. The left column is bosonic ^{7}Li, and the right column is fermionic ^{6}Li. It is clear that these two statistics give rise to very different results at extremely low temperatures.

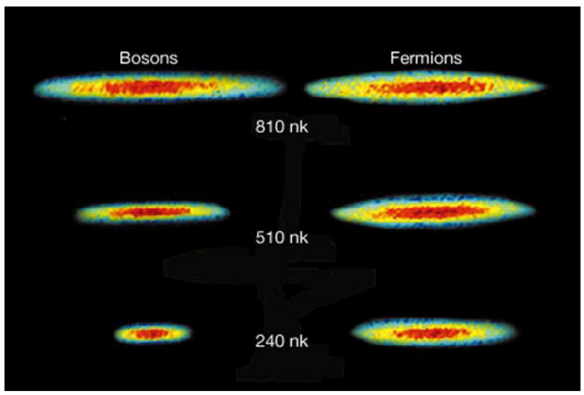

Figure 1: Demonstration of BEC and Fermi degenerate pressure

Hence, creating a BEC is relatively simple in principle — making a extremely cold gas. In most situations, however, the quantum statistical effect would simply be precluded by the more familiar transition to a liquid or solid. Only at extremely low densities, about one-hundred-thousandth (10^{-5}) the density of normal air, can this more conventional transition be avoided. The reason for this lies in the fact that the rate of three-body collisions, which are responsible for forming a cluster or molecule, is proportional to the square of the density. Under such low densities, the formation time of a cluster can be as long as seconds or minutes. While the rate of two-body collisions is only proportional to the density, hence they are more frequent. In the current experiments of alkali gases, the gases can equilibrate within about 10ms by the two-body collisions. So the quantum degeneracy can be achieved in a *metastable* gas phase. However, from the relation, interatomic spacing $\sim \lambda_{dB} \sim 1/\sqrt{T}$, such low densities require the BEC transition temperature to be about nanokelvin (10^{-9}K). In the next section, we will briefly discuss how to obtain such low temperatures.

2 Experiments — Race for Low Temperatures

In the last century, physicists have invented many ingenious devices to obtain temperatures as low as possible in their laboratories. Each advance towards absolute zero temperature has been rewarded with much newer and richer physics. The approach into kelvin (K) regime resulted the discovery of superconductivity in 1911 and of superfluidity of ^{4}He in 1938. Cooling into the millikelvin (10^{-3}K) region revealed the superfluidity of ^{3}He in 1972. In 1980s, laser cooling techinque generated samples of dilute atomic cloud at temperatures of microkelvin (10^{-6}K). In 1995 the atomic BECs were finally realized, which were at temperatures of nanokelvin (10^{-9}K). Each of these major advances in cooling techniques has been an important achievement, and recognized as a Nobel prize in physics.

Currently, temperatures below microkelvin are obtained in two steps. The atomic gas is precooled by the *laser cooling* technique and hence confined in a magnetic trap[1]. In the second step, the *forced evaporative* cooling technique is applied[2] so that the most energetic atoms will escape from the trap while the remaining atoms will thermalize into lower temperatures. Notice in this step the depth of the trap is reduced and it is a irreversible process. The full cooling procedure takes from a few seconds to several minutes. Most systems exhibiting BECs are at temperatures between 500nK and 2μK, with densities between 10^{14} cm^{-3} and 10^{15}cm^{-3}. The total number of atoms ranges from a few hundred to about 10^9. The largest condensates up to now are 30 million atoms in Na and a billion in H.

An important feature of the trapped Bose gases is that they are inhomogeneous and finite with respect to homogeneous and infinite systems discussed in the textbook examples. In most cases, the confining traps can be well approximated as harmonic potentials, which will be discussed in more details in the next section. The trapping frequency ω_{ho} characterizing the harmonic potential provides a natural length scale for the system, $a_{ho} = (\hbar/(m\omega_{ho}))^{1/2}$. It is on the order of a few microns (μm) in current experiments. There are density variations on this scale.

One important consequence of this inhomogeneity is that BEC in trapped gases shows up not only in the *momentum* space, as in the case of superfluid ^{4}He, but also in the *coordinate* space. Another important consequence is the enhanced effect of two-body interaction, as will be shown in the next section. The key point is that, despite the dilute nature of these gases, combination of BEC and harmonic potential greatly enhances the interactions between atoms on measurable quantities.

3 Theoretical Descriptions

3.1 Ideal Bose Gas in a Harmonic Trap

We can safely approximate the confining potential due to the magnetic trap as a harmonic form,

$$V_{ext}(\mathbf{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{1}$$

In the first step of the theoretical discriptions, let us neglect the interactions between the atoms. Thus the many-body Hamiltonian is the sum of single-particle Hamiltonians whose eigenvalues take the form

$$E_{n_x n_y n_z} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z, \tag{2}$$

where $\{n_x, n_y, n_z\}$ are non-negative integers. With this spectrum given, we can get all the other information. For instance, the ground state is obtained by putting all the atoms in the lowest state with $n_x = n_y = n_z = 0$, namely $\Phi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \prod_i \phi_0(\mathbf{r}_i)$, where

$$\phi_0(\mathbf{r}) = \left(\frac{m\omega_{ho}}{\pi\hbar}\right)^{3/4} \exp[-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)], \tag{3}$$

and

$$\omega_{ho} \equiv (\omega_x \omega_y \omega_z)^{1/3}. \tag{4}$$

The density distribution, $n(\mathbf{r}) = N|\phi_0(\mathbf{r})|^2$, grows linearly with N, which is the total number of atoms. The size of the condensate, a_{ho}, is *fixed* by the harmonic potential,

$$a_{ho} = (\hbar/(m\omega_{ho}))^{1/2}. \tag{5}$$

Since the ground state wave function $\phi_0(\mathbf{r})$ is a Gaussian, the momentum distribution of the atoms, which is the Fourier transformation of $\phi_0(\mathbf{r})$, is also a Gaussian centered at zero momentum. Thus this simple model shows peaks in both coordinate and momentum space, as stated in the end of Sec. 2. The distributions in both momentum and coordinate space have indeed been shown to have a sharp peak from experiments[3, 4]. The shape of the peak is certainly different from this simple model due to two-body interactions, which will be discussed later.

We can also calculate the properties of the gas at *finite* temperatures. In the grand canonical ensemble, the total number of particles is given by

$$N = \sum_{n_x, n_y, n_z} \{\exp[\beta(E_{n_x n_y n_z} - \mu)] - 1\}^{-1}, \tag{6}$$

and the total energy is

$$E = \sum_{n_x, n_y, n_z} E_{n_x n_y n_z} \{\exp[\beta(E_{n_x n_y n_z} - \mu)] - 1\}^{-1}, \tag{7}$$

where μ is the chemical potential and $\beta = 1/(K_B T)$. The lowest energy will become macroscopically occupied below the BEC transition temperature T_c^0. When T is less than T_c^0, the chemical potential μ becomes the same as the lowest energy $\frac{\hbar}{2}(\omega_x + \omega_y + \omega_z)$ in the thermodynamic limit. (We will see a proper definition for the thermodynamic limit below.) It is then convenient to seperate out the lowest eigenstate E_{000} from the sum in Eq. (6) and call N_0 the number of particles in this state. Rewrite Eq. (6) as

$$N - N_0 = \sum_{n_x, n_y, n_z \neq 0} \frac{1}{\exp[\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)] - 1} \tag{8}$$

In the semiclassical approximation, $k_B T \gg \hbar\omega_{ho}$, we can replace the sum by an integral

$$N - N_0 = \int\int\int_0^\infty \frac{dn_x dn_y dn_z}{\exp[\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)] - 1}. \tag{9}$$

This approximation is legitimate as we shall see below. After the integration, we obtain

$$N - N_0 = \zeta(3)\left(\frac{k_B T}{\hbar\omega_{ho}}\right)^3, \tag{10}$$

where $\zeta(n)$ is the Riemann zeta function. From this we can find the BEC transition temperature by setting $N_0 \to 0$,

$$k_B T_c^0 = \hbar\omega_{ho}\left(\frac{N}{\zeta(3)}\right)^{1/3} \approx 0.94\ \hbar\omega_{ho} N^{1/3}. \tag{11}$$

From Eq. (11), it is clear that the proper *thermodynamic limit* is obtained by letting $N \to \infty$ and $\omega_{ho} \to 0$, while keeping the product $N\omega_{ho}^3$ constant. Notice this is *different* from the usual definition of thermodynamic limit. We can also get the temperature dependence of the condensate fraction for $T \leq T_c^0$,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^3. \tag{12}$$

These results can be compared with the theory for a uniform Bose gas [5]. In this case, the eigenstates are plane waves with energy $E = \frac{p^2}{2m}$. The sum in Eq. (6) gives $k_B T_c^0 = (2\pi\hbar^2/m)[n/\zeta(3/2)]^{2/3}$, with $n \equiv \frac{N}{V}$, the number density of the system, and $\frac{N_0}{N} = 1 - (\frac{T}{T_c^0})^{3/2}$.

We can see that there are *two* energy scales for an ideal Bose gas in a harmonic trap, the average energy level spacing $\hbar\omega_{ho}$, and the transition temperature $k_B T_c^0$. From Eq. (11), it's easy to see that $k_B T_c^0$ can be much larger than $\hbar\omega_{ho}$. From the discussion in Sec. 2, N ranges from a few hundred to 10^9. This means that the semiclassical approximation, $k_B T \gg \hbar\omega_{ho}$, is valid. The level spacing $\hbar\omega_{ho}$ is fixed by the trapping potential, which is on the order of a few nK in current experiments. For example, in one of the first experiments realizing atomic BECs[6], the average level spacing was about 9nK, corresponding to a critical temperature about 300nK with 40000 atoms in the trap.

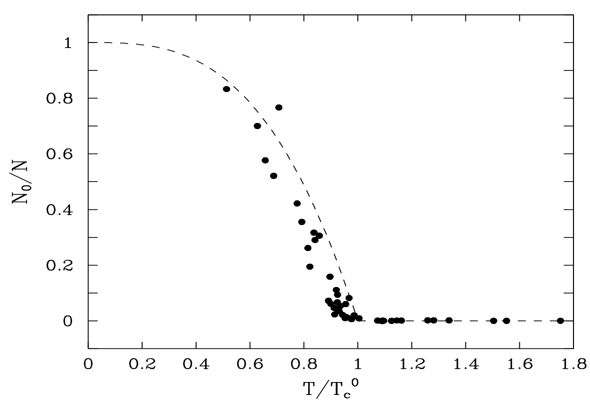

Figure 2: Condensate fraction, N_0/N, as a function of T/T_c^0. Dots are the experimental results[6], while the dashed line is Eq. (12).

This simple model has guided experimentalists to the correct value for the transition temperature. As shown in Fig. 2, the measured transition temperature was actually found to be very close to the ideal Bose gas value predicted by Eq. (11). The condensated fraction below the transition temperature is also close to the form of Eq. (12).

A more quantitative comparison between theory and experiment requires considering two important factors: finite-size effect and interactions between atoms. The role of interactions will be discussed later. Here we briefly discuss the finite-size corrections. In reality, the experiments are always carried out with a finite system, hence the thermodynamic limit is never reached. Thus the Bose-Einstein condensation is not, strictly speaking, a phase transition. It is rounded due to finite-size effect. But this effect is so small that we can still use the word transition temperature to discuss the system just as the experimental results have shown in Fig. 2 that around T_c^0 there *is* a sudden transition. On the other hand, the finite-size effect does change the curve of the condensate fraction. To the lowest order, the result for N_0/N is

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^3 - \frac{3\tilde{\omega}\zeta(2)}{2\omega_{ho}[\zeta(3)]^{2/3}}\left(\frac{T}{T_c^0}\right)^2 N^{-1/3}, \tag{13}$$

where $\tilde{\omega} = (\omega_x + \omega_y + \omega_z)/3$. This is done by considering the large N expansion of the sum in Eq. (6), instead of taking N to be infinity. The corrected curve is shown in Fig. 3. We can see that the finite-size effect reduces the condensate fraction compared to thermodynamic limit.

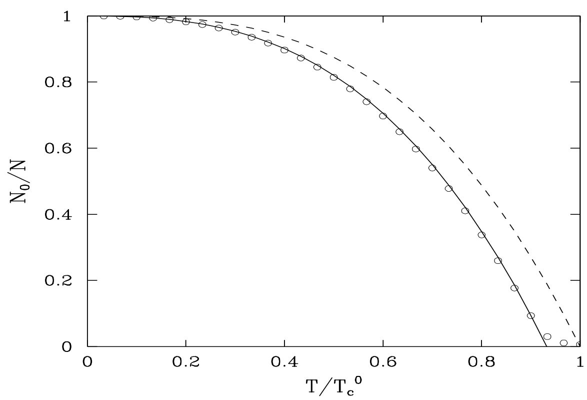

Figure 3: Condensate fraction, N_0/N, as a function of T/T_c^0. Circles are the numerical calculation for $N = 1000$ in Eq. (6). The solid line is Eq. (13), while the dashed line is Eq. (12) corresponding to the thermodynamic limit.

3.2 Effects of Interactions

To discuss interacting systems, we had better use the language of second quantization. In this formalism, the many-body Hamiltonian of N interacting bosons under an external potential is

$$\hat{H} = \int d\mathbf{r}\ \hat{\Psi}^\dagger(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r})\right]\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\ \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r}), \tag{14}$$

where $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r})$ are the bosonic field operators that annihilate and create a particle at position $\mathbf{r}$ respectively, and $V(\mathbf{r}-\mathbf{r}')$ is the two-body interaction potential.

We can start from this Hamiltonian to work out the thermodynamic properties of this interacting system. But the direct calculation method becomes impossible in practice when we have very large N. Instead, we will develop a theoretical description based on the *mean-field* approximation. The key step is to write the bosonic field operator as a sum of two pieces,

$$\hat{\Psi}(\mathbf{r},t) = \Phi(\mathbf{r},t) + \hat{\Psi}'(\mathbf{r},t) \ , \tag{15}$$

where $\Phi(\mathbf{r},t)$ is a complex function, which is the expectation value of $\hat{\Psi}(\mathbf{r},t)$, i.e., $\Phi(\mathbf{r},t) \equiv \langle\hat{\Psi}(\mathbf{r},t)\rangle$. So the modulus of $\Phi(\mathbf{r},t)$ gives the condensate density $n_0(\mathbf{r},t) = |\Phi(\mathbf{r},t)|^2$. Notice the function $\Phi(\mathbf{r},t)$ is not a field operator, instead it is a classical field with the meaning of an *order parameter field.* When the depletion of the condensate is relatively small, i.e., $\hat{\Psi}'$ small, the decomposition of $\hat{\Psi}$ becomes useful. We can then derive an equation for the order parameter $\hat{\Phi}$ by expanding the full equation to the lowest order in $\hat{\Psi}'$. The full equation satisfied by $\hat{\Psi}(\mathbf{r},t)$ is given by the Heisenberg equation

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t) &= [\hat{\Psi},\hat{H}] \\ &= \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + \int d\mathbf{r}'\ \hat{\Psi}^\dagger(\mathbf{r}',t)V(\mathbf{r}'-\mathbf{r})\hat{\Psi}(\mathbf{r}',t)\right]\hat{\Psi}(\mathbf{r},t)\ . \end{aligned} \tag{16}$$

For a *dilute* and *cold* gas, only two-body collisions at low energies are relevant. These collisions can be characterized by the s-wave scattering length a. So we can replace $V(\mathbf{r}'-\mathbf{r})$ in (16) by an effective interaction

$$V(\mathbf{r}'-\mathbf{r}) = g\delta(\mathbf{r}'-\mathbf{r}), \tag{17}$$

where g is determined by the s-wave scattering length a,

$$g = \frac{4\pi\hbar^2 a}{m}\ . \tag{18}$$

Then to the lowest order, we can obtain the desired equation for Φ,

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2\right)\Phi(\mathbf{r},t)\ . \tag{19}$$

This is the well-known Gross-Pitaevskii (GP) equation. Its validity is based on *two* conditions that the s-wave scattering length should be much smaller than the average distance between atoms and that the number of atoms in the condensate should be much larger than one. Let us take a look at the first condition in current experiments. For ^{23}Na $a = 2.75$nm and for ^{87}Rb $a = 5.77$nm. The average density $\bar{n}$ ranges from 10^{14}cm^{-3} to 10^{15}cm^{-3}, as discussed in Sec. 2. Hence $\bar{n}|a|^3$ is always less then 10^{-3}, and the first condition is satisfied.

The GP Eq. (19) can be solved numerically, to compared the solution with the experiment. Fig. 4 is such a comparison plotting the density distribution of 80000 Na atoms [7] as a function of the axial coordinate. The experimental results agree very well with the solid line, the solution of GP Eq. (19). The dashed line is the behavior of

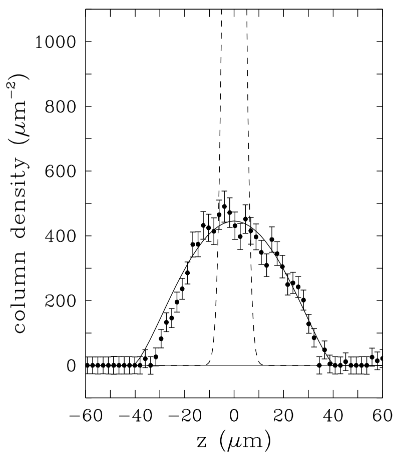

Figure 4: Density distribution of 80000 Na atoms[7] as a function of the axial coordinate. The solid line is the numerical solution of the GP Eq. (19), while the dashed line corresponds to a non-interacting gas.

non-interacting bosons, which is a Gaussian distribution as discussed in Sec. 3.1. This figure shows clearly the role of two-body interaction in reducing the central density and enlarging the size of the cloud, as stated in the end of Sec. 2.

We can also use the GP Eq. (19) to calculate the dynamic properties of the system. In the low temperature limit, the excited states can be found by the linearized GP Eq. (19), i.e., we look for the solution of the form

$$\Phi(\mathbf{r}, t) = e^{-i\mu t/\hbar}\left[\phi(\mathbf{r}) + u(\mathbf{r})e^{-i\omega t} + v^*(\mathbf{r})e^{i\omega t}\right] \tag{20}$$

By keeping only the linear terms, GP Eq. becomes

$$\hbar\omega u(\mathbf{r}) = [H_0 - \mu + 2g\phi^2(\mathbf{r})]u(\mathbf{r}) + g\phi^2(\mathbf{r})v(\mathbf{r}) \tag{21}$$

$$-\hbar\omega v(\mathbf{r}) = [H_0 - \mu + 2g\phi^2(\mathbf{r})]v(\mathbf{r}) + g\phi^2(\mathbf{r})u(\mathbf{r}) \ , \tag{22}$$

where $H_0 = -(\hbar^2/2m)\nabla^2 + V_{\text{ext}}(\mathbf{r})$. From these equations, we can calculate the excitation energies.

For a *uniform* Bose gas, the result for the despersion relation is the famous Bogoliubov form,

$$(\hbar\omega)^2 = \left(\frac{\hbar^2 q^2}{2m}\right)\left(\frac{\hbar^2 q^2}{2m} + 2gn\right) \tag{23}$$

where $\mathbf{q}$ is the wavevector of the excitation and $n = |\phi|^2$ is the density of the gas.

For the case of harmonic potential, the situation is much more complicated. We just show an example in Fig. 5 to illustrate the idea. Points are taken from the experiment[8], while solid lines are the predictions of Eqs. (21) and (22). Different m corresponds to different modes. The important thing is that the theory agree with the experiment.

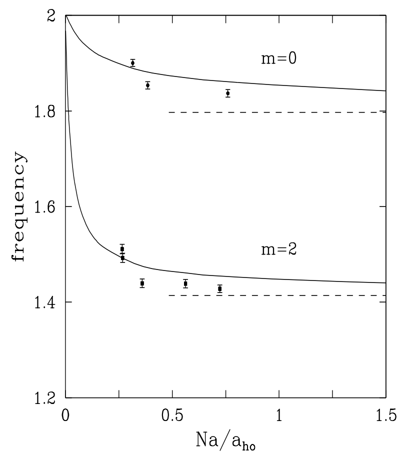

Figure 5: Frequency of the lowest collective modes for N Rb atoms[8].

4 Conclusion and Outlook

We find that there is a general agreement between the mean-field theory and experiments and the two-body interaction plays an importan role.

On the other hand, we only touched some basic ideas of BEC. Many interesting phenomena are untouched, such as superfluidity in BEC, vortices, BEC in lower dimensions, Josephson effect, attactive interactions, "atomic laser", etc. Interested readers can find more in the excellent reviews[9, 10, 11, 12].

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