THE EMERGENCE OF SPACETIME IN THE IKKT MODEL

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ABSTRACT. Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT) have proposed a model describing type-IIB string theory on a flat background. Beginning with the Schild gauge-fixed form of the string action, they found that the matrix regularization of the action led to a zero dimensional matrix model equivalent to the complete dimensional reduction of ten dimensional super Yang Mills theory. By examining an effective action of the matrix model around a diagonal background, an interpretation of the background as spacetime points emerges.
1. A Brief Summary of Matrix Theory and the BFSS Conjecture

What is a matrix theory? Simply put, it is a quantum mechanics with matrix degrees of freedom. It is in general comprised of some $N \times N$ bosonic and fermionic matrices. Matrix theories have the attractive quality that they are not quantum field theories, and thus have none of the peculiarities involved with QFTs, such as renormalization. Indeed, since $N$ is finite (though taken to be large) there are only a finite number of degrees of freedom, so the theory is well defined [Tay01].

In the mid-90’s, Witten found that the low-energy Lagrangian which describes a system of $N$ type-IIA D0-branes is equivalent to the dimensional reduction (the theory on a point) of ten dimensional super Yang-Mills theory to 0+1 dimensions [Wit96]. The dimensionally reduced Hamiltonian is

$$H = \frac{R}{2} \text{Tr} \left\{ P^i P^i - \frac{1}{2} [X^i, X^j][X^i, X^j] + i \theta \gamma_i [X^i, \theta] \right\},$$

where $X^i$ and $P^i$ are $N \times N$ with bosonic entries, and $\theta$ is a set of $N \times N$ matrices with fermionic entries. The number of $\theta$ matrices depends on the type of spin representation $\theta$ is in.

Soon after Witten’s discovery, Banks, Fischler, Shenker, and Susskind (BFSS) conjectured that this Hamiltonian precisely describes M-theory in the light-front coordinate system (again in the limit $N \to \infty$) [BFSS97]. In support of their conjecture, BFSS noted that as expected from supergravity calculations, their model exhibits long-range interactions between gravitons (in this case described by D0-branes) and contains multi-particle states [Tay01].

2. IKKT Model

The IKKT model is one proposed non-perturbative definition of string theory [IKKT97]. It is sometimes referred to as the IIB matrix model, since it is related to a gauge-fixed
form of the Green-Schwarz action for the IIB string.

There are two ways of defining the model. One is as the complete dimensional reduction (to 0-dimensions) of $\mathcal{N} = 1$ super Yang Mills theory. The reduced action is

$$ S = -\frac{1}{g^2} \text{Tr} \left\{ \frac{1}{4} \sum_{\mu,\nu=0}^{9} [A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2} \sum_{\mu=0}^{9} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right\}. $$

The action (2) looks very similar to the dimensionally reduced Hamiltonian of the BFSS conjecture, Eq. (1). This is not just coincidence, they are in fact related by a T-duality.

The action still has $SO(9,1)$ symmetry, as well as the $SU(N)$ gauge symmetry. The $A_\mu$ are $N \times N$ hermitian matrix components of a vector in an $SO(9,1)$ representation. The fermion $\psi$ is in a Majorana-Weyl representation of $\text{Spin}(9,1)$ and has $N \times N$ hermitian matrices for components. The coupling constant $g$ may be absorbed into the fields by a trivial redefinition $[\text{AKT99}].$

The partition function for the theory is then defined via euclideanization of the action (Wick rotation of $A_0$ and $\Gamma^0$);

$$ Z = \int dA d\psi e^{-S_E}. $$

Since there is no notion of time, there are no dynamics to speak of in the model. All information is contained in the partition function as defined above.

The model may also be defined in terms of a grand-canonical partition function

$$ Z[\beta] = \sum_{N=1}^{\infty} \int dX d\Psi e^{S_E[\beta]}. $$

Here $\beta$ is interpreted as a chemical potential dual to the matrix size $N$, and the action is

$$ S_E^{(2)}[\beta] = \frac{1}{2\alpha'^2 \beta} \text{Tr} \left\{ \frac{1}{4} [A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2} \bar{\Psi} \Gamma^\mu [X_\mu, \Psi] \right\} + \beta N. $$

From the form of the actions (2) and (5), it is clear that the action is minimized if the bosonic matrices commute. If that is the case, they may all be simultaneously diagonalized and decomposed into a diagonal part $X_\mu$ and an off-diagonal part $\tilde{A}_\mu$;
\[ A_\mu = X_\mu + \tilde{A}_\mu = \begin{pmatrix} x_\mu^1 \\ \vdots \\ x_\mu^N \end{pmatrix} + \tilde{A}_\mu. \]

The fermion \( \psi \) is also decomposed into diagonal and off-diagonal components \( \xi \) and \( \tilde{\psi} \);

\[
\psi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} + \tilde{\psi}.
\]

Under this transformation, the off-diagonal terms \( \tilde{A}_\mu \) and \( \tilde{\psi} \) become massive. They may then be integrated out, and an effective action for the diagonal bosons (to be interpreted as space-time points) may be obtained by integrating out the diagonal fermions \( \xi^i \).

\[
\int dA d\psi \exp\{-S[A,\psi]\} \longrightarrow \int dX \exp\{-S_{\text{eff}}[X]\}.
\]

Ishibashi et al. interpret the diagonal matrix elements as points of spacetime. By expanding the action as a perturbative series and integrating out all the other fields, an effective action for the spacetime points is found.

The perturbative expansion collects powers of the coupling constant \( g^2 \). This scheme is valid if the spacetime points are widely separated (\(|x^i - x^j| \gg \sqrt{g}\)) \cite{AKT99}. Residual gauge symmetries may be handled by introducing Faddeev-Popov ghosts and gauge-fixing terms. This calculation is carried out, and Feynman diagrams are provided, in \$2.1\) of \cite{AIK98}. To second order in \( g^2 \),

\[
\int [d\tilde{A}d\tilde{\psi}d\bar{d}c] e^{-(S_2+S_{g,F})} \prod_{i<j} \det_{\mu\nu} \left[ \eta_{\mu\nu} + (\xi^i_\mu - \xi^j_\mu)\Gamma_{\mu\nu} (x^i - x^j)(x^i_\nu - x^j_\nu) \right] = \exp -S_{\text{eff}}^{1\text{-loop}}[X,\xi].
\]

Here \( \Gamma^{\mu\nu} = \Gamma^{[\mu}\Gamma^{\alpha}\Gamma^{\nu]} \) are antisymmetrized Dirac matrices in ten dimensions, and \( \eta^{\mu\nu} \) is the original metric on the ten dimensional tangent bundle. To find the effective action, the determinant is expanded;

\[
-S_{\text{eff}}^{1\text{-loop}}[X,\xi] = \sum_{i<j} \text{tr} \left[ \ln \left( \eta^{\mu\nu} + S^{\mu\nu}_{(ij)} \right) \right]
\]

\[
= - \sum_{i<j} \text{tr} \left[ \frac{1}{2} S^2_{(ij)} + \frac{1}{4} S^4_{(ij)} + \frac{1}{6} S^6_{(ij)} + \frac{1}{8} S^8_{(ij)} \right].
\]
Here $S_{(ij)}^{\mu \nu} = (\xi^i_{\mu} - \xi^j_{\mu}) \Gamma^{\mu \alpha \nu} \frac{x^i_{\alpha} - x^j_{\alpha}}{(x^i - x^j)^4} (\xi^i_{\nu} - \xi^j_{\nu})$, and tr means trace over the Lorentz indices. The series terminates at the 8th power since $\xi$ is a 16-component spinor. Due to special cancellation in ten dimensions, $S^2$ and $S^6$ both vanish identically [AKT99].

Finally, the $\xi$ integrations need to be performed in order to find final effective action for the spacetime points. The exponential of the effective action (6) is expanded into a product over all possible pairs of color indices;

$$
\int dX d\xi \prod_{i<j} \left\{ 1 + \frac{1}{4} \text{tr} S^4_{(ij)} + \left[ \frac{1}{2} \left( \frac{1}{4} \text{tr} S^4_{(ij)} \right)^2 + \frac{1}{8} \text{tr} S^8_{(ij)} \right] \right\}.
$$

Aoki et al. [AKT99] refer to the term in square brackets in equation (7) as a “16-fold bond”, since it contains 16 fermionic components ($8 \xi + 8 \bar{\xi}$ terms). The other trace term is referred to as an “8-fold bond” for similar reasons.

The $d\xi$ in the measure of the path integral consists of $16 \times N \, d\xi$ terms. Of course, for the integral to be non-zero they must be saturated. Thus, the fermionic integrals may be reinterpreted as a sum over all ways to combine the 16- and 8-fold bonds into the necessary $16 \times N$ terms.

The bonds are associated with functions of $x^i - x^j$, so they are to be visualized as connecting the spacetime points $x^i$ and $x^j$. Hence, the sum over ways to combine the bonds is in fact a sum over all possible graphs of 8- and 16-fold bonds connecting the points $x^i$.

$$
\int dX d\xi \exp \left\{ -S_{\text{eff}}^{\text{1-loop}}[X, \xi] \right\} = \int dX d\xi \sum_{\text{graphs } G} \prod_{(i,j)^{\text{th}} \text{ bond of } G} \left\{ \frac{1}{4} \text{tr} S^4_{(ij)} \quad \text{or} \quad \left[ \frac{1}{2} \left( \frac{1}{4} \text{tr} S^4_{(ij)} \right)^2 + \frac{1}{8} \text{tr} S^8_{(ij)} \right] \right\}
$$

The first factor is used if the $(i, j)^{\text{th}}$ bond of a graph is 8-fold, while the second is used if it is 16-fold. By examining the effect of first integrating one component of one spinor, it is found that the only graphs which need to be considered are those which can be expressed as a superposition of “maximal trees”; those graphs which do not contain loops [AIK+98].
3. Emergence of spacetime structure

The first hint of the emergence of four dimensional spacetime can be glimpsed by examining a much simpler model; the complete dimensional reduction of four dimensional super Yang Mills. In this case, there is only one bond possible, and the factor corresponding to it is

\[ \text{tr} \left( S_{(ij)}^2 \right) = \delta^{(4)}(\xi^i - \xi^j) \frac{1}{(x^i - x^j)^6}. \]

Thus, in the four dimensional model the effective action for spacetime points becomes

\[ \int dX \exp \left\{ -S_{\text{eff}}^{1\text{-loop}}[X] \right\} = \int dX \sum_{\text{maximal trees } G} \prod_{(i,j) \text{th bond of } G} \frac{1}{(x^i - x^j)^6}, \]

which is a branched-polymer type distribution\cite{AIK98}. The Hausdorff dimension of such a distribution is well-known to be four.

3.1. Hard-core interactions and the 10d model. What if some of the spacetime points are not well separated? Aoki et al. find that if there is one pair of coordinates which do not satisfy \( |x^i - x^j| \gg \sqrt{g} \), but all other pairs satisfy the constraint, then the nearby pair becomes an \( SU(2) \) degree of freedom\cite{AIK98}. This induces a repulsive potential of strength \( -8 \ln r \), with \( r \) the relative coordinate of the two points. In fact, the repulsive pairing calculation generalizes to any number of close pairs, as long as they have well-separated center of mass coordinates. These considerations seem to indicate that the IKKT model features a certain “incompressibility” of the eigenvalue distribution.

This “incompressibility” is factored into the model via the introduction of an additional term to the action,

\[ S_{\text{core}}[X] = \sum_{i<j} \begin{cases} -4 \ln((x^i - x^j)^2/g) & \text{for } (x^i - x^j)^2 \ll g \\ 0 & \text{for } (x^i - x^j)^2 \gg g \end{cases} \]

Although the action isn’t exact for short distance, it is assumed to be in the same universality class as the full model\cite{AIK98}.

Before, it was stated that the four dimensional model exhibited a branched-polymer phase. In fact, if the hard-core interaction is taken into account and the 8-bond interaction terms are ignored in the full ten dimensional theory, then the remaining 16-fold bonds induce a branched-polymer, with a factor of \( 1/(x^i - x^j)^{24} \).

\[ \int dX \exp \left\{ -S_{\text{eff}}^{1\text{-loop}}[X] + S_{\text{core}}[X] \right\} = \int dX e^{S_{\text{core}}[X]} \sum_{\text{maximal trees } G} \prod_{(i,j) \text{th bond of } G} \frac{1}{(x^i - x^j)^{24}} \]
Aoki et al. claim that it is conceivable that the introduction of 8-fold bonds results in the emergence of smooth, four dimensional spacetime. This concept is demonstrated in the double-tree model.

3.1.1. **Double-tree model.** The double-tree model is much simpler than the full set of fermionic integrals. It is a tractable model which hopefully captures the essence of the maximal tree result with 8-fold bonds included. It involves two independent maximal trees which have \( N \) points maximally connected with coordinate \( x^i_\mu \). If the two trees share a bond, it is regarded as 16-fold and assigned an amplitude \( ((x^i - x^j)^2 + g)^{-12} \). All other bonds are assigned \( ((x^i - x^j)^2 + g)^{-6} \). This last factor is formulated as to average over the complicated orientation dependence of the 8-fold bonds\(^{[AKT99]}\).

In fact, the averaging process suppresses the formation of 8-fold bonds. To account for this, a factor of \( e^{-\lambda/2} \) may be introduced. Then,

\[
\int dX e^{-S_{\text{core}}} d\xi e^{-S_{\text{eff}}}[X] \rightarrow \int dX e^{-S_{\text{core}}} \sum_{2 \text{ maximal trees} G} \prod_{(i,j)^{\text{th}} \text{ bond of } G} \left\{ \left[ ((x^i - x^j)^2 + g)^{-6} e^{-\lambda/2} \right] \text{ or } \left[ ((x^i - x^j)^2 + g)^{-12} \right] \right\}.
\]

Since the IKKT model has no free parameters, \( \lambda \) must be fixed. However, it can be treated as a parameter so that the phase structure may be examined. As \( \lambda \to \infty \), most bonds correspond across the two trees. The distribution behaves as an ordinary branched polymer\(^{[AIK+98]}\).

As \( \lambda \to -\infty \), 8-fold bonds dominate, and it turns out that all of the short distance pairs are located in a volume of finite size, called a droplet. Numerical simulations have shown that there is a phase change between the droplet and branched polymer phases which occurs at positive \( \lambda \). Aoki et al.\(^{[AIK+98]}\) calculated the free energy of the droplet phase, and found it to be a minimum when the droplet state is realized in four dimensions. See figure (1) for dimensional comparison.

3.1.2. **Perturbative Branched Polymer.** Another (albeit related) way to view the generation of four dimensions is to regard the branched-polymer phase as a background, and the addition of 8-fold bonds as perturbations. Such perturbations would presumably compress the system to the Hausdorff dimension of the background, which is four. For details see \(^{[AIK+98]}\).

3.2. **Spontaneous Symmetry Breaking.** The euclideanization of Eq. (2) is manifestly \( SO(10) \) invariant. If we assume that this symmetry is spontaneously broken to a subgroup, say \( SO(d) \), with \( d < 10 \), the free energy of the resulting configuration may be calculated by Gaussian expansion. The complex phase of the Pfaffian (determinant) induces this symmetry breaking, as was shown in a exactly soluble model by Nishimura\(^{[Nis02]}\).
An analysis of the free energy associated with the broken symmetry was performed by Nishimura and Sugino [NS02]. They found that the free energy, to third order in the expansion, was minimized when the subgroup was $SO(4)$.

The effects of the assumption of spontaneous symmetry breaking were also investigated by Kawai et al. [KKK+02]. Numerical techniques such as an Improved Mean Field Approximation (IMFA) and an Improved Taylor Expansion (ITE) were employed to calculate a 2PI free energy for models in which the original $SO(10)$ invariance is broken to $SO(n)$, with $1 \leq n \leq 7$. They too find a minimum of the free energy when $n = 4$. The authors also calculate the ratio of the “size” of the two eigenvalue subdistributions; the unbroken $SO(4)$ subset (called “our space”, $R$), and everything else (“internal space”, $r$).
Figure 2. Free energy contributions of odd orders in perturbation theory *(From [KKK+02])

\[ R^2 = \left\langle \frac{1}{N} \text{Tr} A_i^2 \right\rangle \]

\[ r^2 = \left\langle \frac{1}{N} \text{Tr} A_{i0}^2 \right\rangle \]

The ratio \( \rho = R/r \) was calculated to fifth order for \( SO(7) \) and \( SO(4) \) as the unbroken subgroups; see figure (3). As was hoped, for \( SO(4) \) the 4 dimensional space has a much larger extent than the remaining 6 dimensional space. In fact, adding higher orders in perturbation theory causes \( \rho \) to increase rapidly.

Unfortunately, the authors do not fully understand why their numerical methods work. IMFA is not a true approximation scheme, and they do not know why the free energy appears to converge to its exact value. In their words, “the results [of the IMFA] are magically nice.” [KKK+02]

4. IKKT Problems

IKKT is not without some difficulties. One is that it can only be formulated as a Lagrangian theory; we can only do path integrals. A true Hamiltonian cannot be constructed because there are no time derivatives, and hence no canonical momenta, commutators, etc. Furthermore, this means that no Hilbert space of physical states is possible, which makes it somewhat difficult to describe particles.
Recall that IKKT is related by a T-duality to BFSS, in fact a time-like T-dualities. Time-like T-dualities are a somewhat controversial procedure, although there has been some recent work to put such transformations on a more rigorous footing [Keu04].

5. CONCLUSIONS

The IKKT model strives to provide a purely quantum mechanical description of type IIB string theory. Under some phenomenological assumptions and anzate motivated from other approaches to gravitational study, hints of the emergence of four macroscopic plus six microscopic dimensions of space have been teased out. However, difficulties in calculation and ambiguities of definition cast doubts on the validity of such an attractive result. Much room remains for more detailed and rigorous analysis.
References


