Emergent States of Matter

HOMEWORK SHEET 8

Due 5pm Mon 3rd Dec 2012 in the 569 ESM box.

Question 8–1.

This question concerns spontaneous symmetry breaking in charged superfluids, i.e. superconductors, and is based on the coarse-grained free energy

$$F\{\psi, \vec{A}\} = \int d^dx \left\{ \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + |(\nabla - ie^* \vec{A})\psi|^2 + \frac{(\nabla\times \vec{A})^2}{8\pi} \right\}$$

where e^* is the effective charge of the condensate, which we know to be $e^* = 2e$ from BCS theory, α is proportional to $T - T_c$, ψ is the complex order parameter for the superconducting transition, and $\vec{A}$ is the electromagnetic vector potential. In the absence of an electromagnetic field, the order parameter spontaneously breaks the "U(1) global gauge symmetry" (i.e. the invariance to changes of phase of ψ) when $\alpha(T) < 0$ and $\psi^2 = -\alpha/\beta$. In the following we will chose the phase of ψ to be zero, so that $\psi = v \equiv \sqrt{-\alpha/\beta}$.

(a) Let $\vec{A}$ now be non-zero, and expand ψ about v to second order to find the effective free energy for the system in terms of the real and imaginary parts of the fluctuation in ψ, ψ_1 and ψ_2 respectively. It will be most convenient for you to calculate $\Delta F \equiv F\{\psi, \vec{A}\} - F\{\psi, \vec{0}\}$. Don't forget the gradient terms.

(b) Your resulting expression is difficult to interpret physically because it involves a cross term between a component of the order parameter and the vector potential. Show that this can be removed by making a gauge transformation on the vector potential: $\vec{A} = \vec{A}' + \nabla\Lambda$ where $\vec{A}'$ is the transformed vector potential and Λ is a function that you should determine.

(c) Your resulting expression for ΔF should contain ψ_1 and $\vec{A}'$ only. What happened to ψ_2? Has the number of degrees of freedom changed during the transition?

(d) Explain the physical significance of the $\vec{A}'$ dependence, by writing the Euler-Lagrange equations for ϕ_1 and $\vec{A}'$. Be sure to explain what is the "mass" of all the fields left in the problem. What happened to the Goldstone boson that is present in the neutral superfluid case when $e^* = 0$?

This phenomenon is known as the Anderson-Higgs mechanism and plays an important role in condensed matter physics and higher energy physics.

Question 8–2.

In this question, you will show that it is impossible for a homogeneous superconducting state to be present in a homogeneous magnetic field, but that a periodic superconducting state is possible.

In a constant magnetic field H, the vector potential can be chosen to be $\mathbf{A} = \frac{1}{2}\mathbf{H}\times\mathbf{x}$. Under gauge transformations, the field operators for electrons ψ and the vector potential transform like: $\mathbf{A} \to \mathbf{A} + \nabla\Lambda$ and $\psi \to \psi \exp(ie\Lambda/\hbar c)$ where e is the charge on the electron divided by Planck's constant times the speed of light.

(a) Verify this by considering the Schrodinger equation for a particle in an electromagnetic field, described by the Hamiltonian:

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + e\phi + V$$

Here ϕ is the scalar potential, V is an external potential, and $\mathbf{A}$ is the vector potential. Make the gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\Lambda$ and $\phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$ and show that $\psi \to \psi \times \exp(ie\Lambda/\hbar c)$.

Consider the so-called equal time anomalous Green's function, written in the form

$$F_A(\mathbf{x};\mathbf{y}) = \left\langle \left[\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y}) \right] \right\rangle_A$$

where the average is taken in the presence of the vector potential $\mathbf{A}$. (Hint: Don't worry too much about the spin degrees of freedom in this problem. I've shown them for concreteness, however.)

(b) Show that if we shift the origin of co-ordinates by an arbitrary amount $\mathbf{a}$ (i.e. make a Galilean transformation) then

$$F_A(\mathbf{x} + \mathbf{a}, \mathbf{y} + \mathbf{a}) = e^{i\frac{e}{2}(\mathbf{H}\times\mathbf{a})\cdot(\mathbf{x}+\mathbf{y})} F_A(\mathbf{x}, \mathbf{y}).$$

Hint: See how the vector potential changes by the co-ordinate shift, and then remove that change by making a suitable gauge transformation.

(c) Now make a further arbitrary Galilean transformation, this time by the amount $\mathbf{b}$. Using your answer (or mine!) to (b), write down how the anomalous Green's function transforms.

(d) What would have happened if we had done the shift by $\mathbf{a} + \mathbf{b}$ all in one go? Answer this, and hence show that it is impossible for a uniform superconducting state to exist in a uniform non-zero magnetic field.

(e) If the superconducting state is not uniform, then we can choose the vectors $\mathbf{a}$ and $\mathbf{b}$ to form a periodic lattice in the plane perpendicular to the field $\mathbf{H}$. If we assume that the lattice is a triangular lattice, show that the lattice spacing is $|\mathbf{a}| = (4\pi/\sqrt{3}eH)^{1/2}$.

(f) Our use of ODLRO was essential in this calculation. Outside the superconducting state, F_A would be zero. To see why ODLRO was essential, consider now what would have happened if we had used the regular Green function, i.e. density

$$G_A(\mathbf{x};\mathbf{y}) = \left\langle \left[\psi^+{}_\uparrow(\mathbf{x})\psi_\uparrow(\mathbf{y}) \right] \right\rangle_A$$

Redo part (b) for G_A. You should find that everything is translationally invariant, and so the result we derived does not go through – there is no information that we can use to deduce the structure of the superconducting state.