Emergent States of Matter

HOMEWORK SHEET 7

Due 12 noon Mon 19 Nov 2012 or earlier

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 7–1.

This question concerns the behaviour of correlations in a system with a continuous symmetry at its lower critical dimension. This is the dimension above which ordering is possible for non-zero T. We saw in class that for Bose-Einstein condensation, described by a complex order parameter, $d_c = 2$. Note that a similar more technically complicated analysis applies to smectic liquid crystals, where $d_c = 3$.

(a) Consider a system with a complex order parameter $\psi = S_1 + iS_2$ described by the Hamiltonian

$$-\mathcal{H} = \int d^2r \left[\frac{1}{2}|\nabla\psi|^2 + \frac{u_0}{4}\left(|\psi|^2 - \frac{|r_0|}{u_0}\right)^2 \right]$$

At low temperatures, the amplitude degrees of freedom are frozen out, but the phase fluctuations are strong as we have seen previously. Writing $\psi = A\exp(i\theta(\mathbf{r}))$, where A is the temperature dependent amplitude, which you should determine, show that the effective Hamiltonian is

$$-\mathcal{H} = \frac{K}{2}\int d^2r(\nabla\theta)^2$$

and determine the spin wave stiffness K. The Hamiltonian we have found is that of a 2D XY spin system with effective exchange interaction $J = K(k_BT)$.

(b) Using the fact that the Hamiltonian is Gaussian, show that the normalized order parameter correlation function $G(\mathbf{r}) \equiv \langle\psi^*(\mathbf{r})\psi(0)\rangle / A^2$ obeys

$$G(\mathbf{r}) = \exp\left[-\frac{1}{2}\frac{2k_BT}{J}\int_0^\Lambda \frac{d^2k}{(2\pi)^2}\frac{1 - e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2} \right]$$

where Λ is the coarse-graining scale.

(c) For large $r \gg \Lambda^{-1}$, the oscillatory exponential term can be neglected. By change of variable, and making sure to consider what happens to the limits of integration when doing so, show that the integral can be approximated by its logarithmic divergence, leading to $G(r) = \left(\frac{r}{\Lambda^{-1}}\right)^{-\eta}$, where $\eta = k_BT/2\pi J$

This important result shows that at the lower critical dimension, correlations exhibit power law decay, with a temperature dependent, continuously varying exponent. At higher temperatures, our phase approximation must break down, and at higher temperatures still, we expect the system to exhibit the usual exponential decay of correlations. Hence we conclude that at some intermediate temperature, there is a phase transition between a state with exponential correlations and a state with power law correlations. This is the celebrated Kosterlitz-Thouless transition.

Question 7–2.

(a) Starting from the Hamiltonian for a 2D superfluid

$$\mathcal{H} = \frac{\rho_s}{2}\int v_s^2\, d^2\mathbf{r}$$

calculate the energy of a quantum vortex in a circular 2D condensate of radius R, in terms of the system radius R and core size a and C, the energy/unit area of the region of the core.

(b) Consider two quantum vortices in the 2D condensate, separated by a distance r, with topological charges $q = \pm 1$. Calculate the energy of a pair of vortices, one with sign +1, the other with sign -1, and show that

$$E(r) = 2\pi \rho_s (\hbar/m)^2 \ln(Cr/a)$$

Note that this energy is finite, so vortex pairs can be thermally excited for $T > 0$.

(c) If the vortices are initially placed in an $x - y$ coordinate system with the +1 vortex at $y = r/2$ and the -1 vortex at $y = -r/2$, what is the resulting magnitude and direction of the velocity of the pair?

(d) Suppose now that there is only a single vortex in the system. Estimate the number of ways that the vortex can be placed in the system, and hence calculate the free energy of the single vortex. You should find that

$$F/k_B T = (\pi K - 2) \ln R + \cdots$$

where $K \equiv (\hbar/m)^2 \rho_s / k_B T$. Hence show that as long as we can neglect the interaction between vortices, it is thermodynamically favorable for vortices to proliferate above at critical temperature given implicitly by $K^* = 2/\pi$. This is the Kosterlitz-Thouless transition temperature, and the result obtained here by elementary methods is correct when treated by renormalization group methods.

(e) Using this answer and your results for 7–1, show that correlations decay algebraically at the transition temperature with a 1/4 power law.