Emergent States of Matter

HOMEWORK SHEET 6

Due 10am Tue 24 April 2018 in the 569 box.

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 6–1.

This question concerns the pairing of two fermions at zero temperature: in it you will calculate the characteristic size of a Cooper pair. This calculation does not do the full many-body problem that you did already, but shows you how the essential singularity obtained in the BCS theory has a simple derivation that can be understood from just thinking about wavefunctions. Parts (a)-(d) are done in the online lecture notes, but please try not to look there for help.

Consider two fermions at $T = 0$ in a degenerate fermion fluid, say a nucleus or superconductor, with momenta $\hbar\mathbf{q}$ and $-\hbar\mathbf{q}$, where $|\mathbf{q}| = k_F$ and in this problem the subscript "F" denotes Fermi momentum or energy etc. Their center of mass is at rest. Let the positions be $\mathbf{r}_1$ and $\mathbf{r}_2$. The spatial wavefunction of the pair is given by $\psi(|\mathbf{r}_1 - \mathbf{r}_2|)$ which satisfies the two-particle Schrödinger equation with a potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ and a resulting energy that we will write as $2\epsilon_F + \epsilon$. Thus, the interaction energy of the pair is ϵ, measured from the energy the system would have if V were zero (i.e. twice the Fermi energy). We will see that $\epsilon < 0$, if the other fermions in the system act to implement the Pauli principle, preventing scattering below the Fermi surface. This calculation is inferior to the full BCS many-body calculation you already did, but is instructive in showing you how to get non-perturbative pairing from solving a simple quantum mechanics problem.

Since the overall wavefunction of the fermions is antisymmetric under exchange, and the spatial part is written as a function of $(|\mathbf{r}_1 - \mathbf{r}_2|)$, the spins must be opposite. We'll solve the problem in Fourier space, writing

$$\psi(r) = \sum_k g_k e^{i\mathbf{k}\cdot\mathbf{r}}$$

If V were zero, then the g_k would be zero except for $k = \pm q$. When $V \neq 0$, the pair gets scattered from $\pm q$ to a different pair of wavevectors on the Fermi surface $\pm q'$. We'll implement the Pauli exclusion principle by $g_k = 0$ for $|\mathbf{k}| < k_F$.

(a) Derive an algebraic equation satisfied by all the g_k in terms of the matrix element $V_{kk'} = \int V(\mathbf{r}) \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}]$.

(b) Approximate that $V_{kk'} = -U$ in a thin shell of energy thickness $\Delta E \ll \epsilon_F$ around the Fermi surface. Here $U > 0$. This can be shown to be a reasonable approximation in a superconductor, where this energy is the Debye energy of phonons exchanged between the electrons, leading to an attractive interaction. Show that $g_k = D/[\epsilon + 2\epsilon_F - \hbar^2 k^2/m]$ where the constant $D = -U \sum_k g_k$ and the last sum is over the shell around the Fermi surface.

(c) Hence show that the pair energy satisfies an analogue of the BCS gap equation derived previously:

$$1 = U \int_0^{\Delta E} \frac{N(\xi)\, d\xi}{2\xi - \epsilon}$$

where $\xi \equiv \hbar^2 k^2/2m - \epsilon_F$ and N is the density of states, well approximated by its value $N(0)$ at the Fermi surface as volume $\times m k_F / 2\pi^2\hbar^2$.

(d) Hence show that for $UN(0) \ll 1$ the fermions can pair up with an energy

$$\epsilon = -2\Delta E \exp(-2/N(0)U).$$

This is surprising, because for any arbitrarily small U, we predict is a bound state.

(e) Starting from the formula for the radius of a Cooper pair, $R_c^2 \equiv \langle R^2 \rangle$, where the expectation value is taken with respect to the wavefunction calculated above, and $\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2$ is the separation of the Cooper pairs, show that

$$R_c^2 = \frac{\sum_k |\partial_k g_k|^2}{\sum_k |g_k|^2}.$$

(f) Using the result for $g_k = D/(\epsilon - 2\xi)$, show that $R_c = A\hbar v_F/\epsilon$ and determine the constant A.

Hint: Make the approximations that $N(\xi) \approx N(0)$ and choose the limits on the integrals that you do in a judicious way. Justify the approximations that you make.

This last calculation is close to that given originally by Cooper, but is in fact not quite correct for a rather subtle reason. Can you spot the error? (Don't waste a lot of time on this!).

Question 6–2.

In this question, you will show that it is impossible for a homogeneous superconducting state to be present in a homogeneous magnetic field, but that a periodic superconducting state is possible.

In a constant magnetic field H, the vector potential can be chosen to be $\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{x}$. Under gauge transformations, the field operators for electrons ψ and the vector potential transform like: $\mathbf{A} \to \mathbf{A} + \nabla\Lambda$ and $\psi \to \psi \exp(ie\Lambda/\hbar c)$ where e is the charge on the electron divided by Planck's constant times the speed of light.

(a) Verify this by considering the Schrodinger equation for a particle in an electromagnetic field, described by the Hamiltonian:

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + e\phi + V$$

Here ϕ is the scalar potential, V is an external potential, and $\mathbf{A}$ is the vector potential. Make the gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\Lambda$ and $\phi \to \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$ and show that $\psi \to \psi \times \exp(ie\Lambda/\hbar c)$.

For the rest of this problem, please use units where $\hbar = c = 1$.

Consider the so-called equal time anomalous Green's function, written in the form

$$F_A(\mathbf{x};\mathbf{y}) = \left\langle \left[\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y}) \right] \right\rangle_A$$

where the thermal average is taken in the presence of the vector potential $\mathbf{A}$. (*Hint: Don't worry too much about the spin degrees of freedom in this problem. I've shown them for concreteness, however. Also, don't worry too much about how you would evaluate the anomalous Green's function in practice, by functional integration or anything like that. You do not need to know how to calculate it explicitly from a particular Hamiltonian in order to do this question. That is because we are looking at the symmetry properties of this object.*)

(b) Show that if we shift the origin of co-ordinates by an arbitrary amount $\mathbf{a}$ (i.e. make a Galilean transformation) then

$$F_A(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{a}) = e^{i\frac{e}{2}(\mathbf{H}\times\mathbf{a})\cdot(\mathbf{x}+\mathbf{y})}F_A(\mathbf{x},\mathbf{y}).$$

Hint: See how the vector potential changes by the co-ordinate shift, and then remove that change by making a suitable gauge transformation.

(c) Now make a further arbitrary Galilean transformation, this time by the amount $\mathbf{b}$. Using your answer (or mine!) to (b), write down how the anomalous Green's function transforms.

(d) What would have happened if we had done the shift by $\mathbf{a}+\mathbf{b}$ all in one go? Answer this, and hence show that it is impossible for a uniform superconducting state to exist in a uniform non-zero magnetic field.

(e) If the superconducting state is not uniform, then we can choose the vectors $\mathbf{a}$ and $\mathbf{b}$ to form a periodic lattice in the plane perpendicular to the field $\mathbf{H}$. If we assume that the lattice is a triangular lattice, show that the lattice spacing is $|\mathbf{a}| = (4\pi/\sqrt{3}eH)^{1/2}$.

(f) Our use of ODLRO was essential in this calculation. Outside the superconducting state, F_A would be zero. To see why ODLRO was essential, consider now what would have happened if we had used the regular Green function, i.e. density

$$G_A(\mathbf{x};\mathbf{y}) = \left\langle \left[\psi^+{}_\uparrow(\mathbf{x})\psi_\uparrow(\mathbf{y}) \right] \right\rangle_A$$

Redo part (b) for G_A. You should find that everything is translationally invariant, and so the result we derived does not go through – there is no information that we can use to deduce the structure of the superconducting state.