Emergent States of Matter

HOMEWORK SHEET 6

Due 5pm Mon 12 Nov 2012 in the 569 box.

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 6–1.

This question concerns the pairing of two fermions at zero temperature: in it you will calculate the characteristic size of a Cooper pair. This calculation does not do the full many-body problem that you did already, but shows you how the essential singularity obtained in the BCS theory has a simple derivation that can be understood from just thinking about wavefunctions. Parts (a)-(d) are done in the online lecture notes, but please try not to look there for help.

Consider two fermions at $T = 0$ in a degenerate fermion fluid, say a nucleus or superconductor, with momenta $\hbar\mathbf{q}$ and $-\hbar\mathbf{q}$, where $|\mathbf{q}| = k_F$ and in this problem the subscript "F" denotes Fermi momentum or energy etc. Their center of mass is at rest. Let the positions be $\mathbf{r}_1$ and $\mathbf{r}_2$. The spatial wavefunction of the pair is given by $\psi(|\mathbf{r}_1 - \mathbf{r}_2|)$ which satisfies the two-particle Schrödinger equation with a potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ and a resulting energy that we will write as $2\epsilon_F + \epsilon$. Thus, the interaction energy of the pair is ϵ, measured from the energy the system would have if V were zero (i.e. twice the Fermi energy). We will see that $\epsilon < 0$, if the other fermions in the system act to implement the Pauli principle, preventing scattering below the Fermi surface. This calculation is inferior to the full BCS many-body calculation you already did, but is instructive in showing you how to get non-perturbative pairing from solving a simple quantum mechanics problem.

Since the overall wavefunction of the fermions is antisymmetric under exchange, and the spatial part is written as a function of $(|\mathbf{r}_1 - \mathbf{r}_2|)$, the spins must be opposite. We'll solve the problem in Fourier space, writing

$$\psi(r) = \sum_k g_k e^{i\mathbf{k}\cdot\mathbf{r}}$$

If V were zero, then the g_k would be zero except for $k = \pm q$. When $V \neq 0$, the pair gets scattered from $\pm q$ to a different pair of wavevectors on the Fermi surface $\pm q'$. We'll implement the Pauli exclusion principle by $g_k = 0$ for $|\mathbf{k}| < k_F$.

(a) Derive an algebraic equation satisfied by all the g_k in terms of the matrix element $V_{kk'} = \int V(\mathbf{r}) \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}]$.

(b) Approximate that $V_{kk'} = -U$ in a thin shell of energy thickness $\Delta E \ll \epsilon_F$ around the Fermi surface. Here $U > 0$. This can be shown to be a reasonable approximation in a superconductor, where this energy is the Debye energy of phonons exchanged between the electrons, leading to an attractive interaction. Show that $g_k = D/[\epsilon + 2\epsilon_F - \hbar^2 k^2/m]$ where the constant $D = -U \sum_k g_k$ and the last sum is over the shell around the Fermi surface.

(c) Hence show that the pair energy satisfies an analogue of the BCS gap equation derived previously:

$$1 = V \int_0^{\Delta E} \frac{N(\xi)\, d\xi}{2\xi - \epsilon}$$

where $\xi \equiv \hbar^2 k^2/2m - \epsilon_F$ and N is the density of states, well approximated by its value $N(0)$ at the Fermi surface as volume $\times m k_F/2\pi^2\hbar^2$.

(d) Hence show that the fermions can pair up with an energy

$$\epsilon = -2\Delta E \exp(-2/N(0)U).$$

(e) Starting from the formula for the radius of a Cooper pair, $R_c \equiv \langle R^2 \rangle$, where the expectation value is taken with respect to the wavefunction calculated above, and $\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2$ is the separation of the Cooper pairs, show that

$$R_c^2 = \frac{\sum_k |\partial_k g_k|^2}{\sum_k |g_k|^2}.$$

(f) Using the result for $g_k = D/(\epsilon - 2\xi)$, show that $R_c = A\hbar v_F/\epsilon$ and determine the constant A.
Hint: Make the approximations that $N(\xi) \approx N(0)$ and choose the limits on the integrals that you do in a judicious way. Justify the approximations that you make.

This last calculation is close to that given originally by Cooper, but is in fact not quite correct for a rather subtle reason. Can you spot the error? (Don't waste a lot of time on this!).