Emergent States of Matter

HOMEWORK SHEET 3

Due 5pm Fri 19 Oct 2012 in the 569 ESM box.

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 3–1.
In the two-fluid model of superfluidity, we expand the energy per unit volume E/V as a function of the normal velocity $\mathbf{v}_n$ and the superfluid velocity $\mathbf{v}_s$ to second order. The result is that

$$E/V = \frac{1}{2}\left(\rho_n v_n^2 + \rho_s v_s^2\right) + O(v_n^3, v_s^3)$$

The momentum density (i.e. the total mass current) is given by $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$ and the total density of the superfluid is $\rho = \rho_n + \rho_s$. In this question, you will calculate the behaviour of $\rho_n(T)$ at low temperatures. Our strategy is to calculate the momentum density of the gas of quasi-particles, and equate that to $\rho_n \mathbf{v}_n$.

The total number of quasi-particles with momentum $\mathbf{p}$ in the rest frame of the excitations is given by the usual Bose-Einstein distribution $n(\epsilon(\mathbf{p}))$, where $\epsilon(\mathbf{p})$ is the quasi-particle spectrum. Suppose there is a net current of quasi-particles moving at a given velocity $\mathbf{v}$: then the number of quasi-particles is given by integrating $n(\epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v})$ over momentum. The momentum density is then given by

$$\rho_n \mathbf{v} = \int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{p} n(\epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v}).$$

(a) For $|v|$ small, expand the expression above to first order in $|v|$and hence show that

$$\rho_n = -\frac{1}{3(2\pi\hbar)^3} \int d^3p p^2 \frac{\partial n(\epsilon(|\mathbf{p}|)}{\partial \epsilon}$$

(b) At low temperatures, using the phonon branch of the spectrum, show that

$$\rho_n = A(k_B T)^4$$

and evaluate A in terms of $\hbar$ and the speed of first sound c_1.

Question 3–2.
Calculate the depletion of the condensate $D(T)$, defined in lectures, at small non-zero temperature T. You will need to do the following steps:

(a) Express $D(T) - D(0)$ as an integral involving coherence factors u_k and v_k.
(b) Which region of momentum space does the dominant contribution to the integral come from?
(c) Approximate the integral accordingly, and derive the final result.
[*Hint:* $\int y(e^y - 1)^{-1}\, dy = \pi^2/6$]
(d) Compare your result for the depletion with your result from Q3–1. Comment on any difference.

Question 3–3.

Consider free bosons in two dimensions. In this question, you will review Bose-Einstein condensation, and the analysis you are asked to do will help you understand the absence of ODLRO in 2D, that we discussed in class. I have attached to this problem set my notes on elementary aspects of Bose-Einstein condensation for free bosons in 3D, from my course on statistical mechanics, as a review.

(a) Replace the sum over states by an integral, perform the integral, and hence find the exact closed form expression relating the number of particles N to the fugacity $z \equiv e^{\beta\mu(T)}$.

(b) Hence determine $\mu(T)$ and sketch your result.

(c) Calculate the expected fraction of particles in the ground state and in excited states, and hence find the transition temperature for Bose-Einstein condensation in two dimensions.

(d) The function $b_\nu(z) \equiv \sum_{l=1}^{\infty} z^l/l^\nu$ tends to the Riemann zeta function $\zeta(\nu) \equiv \sum_{l=1}^{\infty} 1/l^\nu$ as $z \to 1$, which is finite for $\nu > 1$. Show that this implies that for $d > 2$ there is Bose-Einstein condensation for $T > 0$.

(e) A theorist's rule of thumb says "For $d \leq 2$, long wavelength fluctuations prevent Bose-Einstein condensation". From your analysis in (d), explain why it can be said that the divergence of $b_\nu(z)$ as $z \to 1$ for $\nu \leq 1$ prevents Bose-Einstein condensation. Illustrate the rule of thumb by writing down the integral for N at $\mu = 0$ in $\mathbf{k}$ space, and show that the contribution of modes near $|\mathbf{k}| = 0$ is divergent for $d \leq 2$.