

## Emergent States of Matter

### HOMEWORK SHEET 3

Due 5pm Mon 22 March 2021 in the 569 ESM box.

*Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.*

#### Question 3–1.

In the two-fluid model of superfluidity, we expand the energy per unit volume  $E/V$  as a function of the normal velocity  $\mathbf{v}_n$  and the superfluid velocity  $\mathbf{v}_s$  to second order. The result is that

$$E/V = \frac{1}{2} (\rho_n v_n^2 + \rho_s v_s^2) + O(v_n^3, v_s^3)$$

The momentum density (i.e. the total mass current) is given by  $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$  and the total density of the superfluid is  $\rho = \rho_n + \rho_s$ . In this question, you will calculate the behaviour of  $\rho_n(T)$  at low temperatures. Our strategy is to calculate the momentum density of the gas of quasi-particles, and equate that to  $\rho_n \mathbf{v}_n$ .

The total number of quasi-particles with momentum  $\mathbf{p}$  in the rest frame of the excitations is given by the usual Bose-Einstein distribution  $n(\epsilon(\mathbf{p}))$ , where  $\epsilon(\mathbf{p})$  is the quasi-particle spectrum. Suppose there is a net current of quasi-particles moving at a given velocity  $\mathbf{v}$ : then the number of quasi-particles is given by integrating  $n(\epsilon(\mathbf{p}) - \mathbf{p} \cdot \mathbf{v})$  over momentum. The momentum density is then given by

$$\rho_n \mathbf{v} = \int \frac{d^3 p}{(2\pi\hbar)^3} \mathbf{p} n(\epsilon(\mathbf{p}) - \mathbf{p} \cdot \mathbf{v}).$$

- (a) For  $|v|$  small, expand the expression above to first order in  $|v|$  and hence show that

$$\rho_n = -\frac{1}{3(2\pi\hbar)^3} \int d^3 p p^2 \frac{\partial n(\epsilon(|\mathbf{p}|))}{\partial \epsilon}$$

- (b) At low temperatures, using the phonon branch of the spectrum, show that

$$\rho_n = A(k_B T)^4$$

and evaluate  $A$  in terms of  $\hbar$  and the speed of first sound  $c_1$ .

#### Question 3–2.

Consider free bosons in two dimensions. In this question, you will review Bose-Einstein condensation, and the analysis you are asked to do will help you understand the absence of ODLRO in 2D, that we discussed in class. I have attached to this problem set my notes on elementary aspects of Bose-Einstein condensation for free bosons in 3D, from my course on statistical mechanics, as a review.

- (a) Replace the sum over states by an integral, perform the integral, and hence find the exact closed form expression relating the number of particles  $N$  to the fugacity  $z \equiv e^{\beta\mu(T)}$ .
- (b) Hence determine  $\mu(T)$  and sketch your result.
- (c) Calculate the expected fraction of particles in the ground state and in excited states, and hence find the transition temperature for Bose-Einstein condensation in two dimensions.
- (d) The function  $b_\nu(z) \equiv \sum_{l=1}^{\infty} z^l / l^\nu$  tends to the Riemann zeta function  $\zeta(\nu) \equiv \sum_{l=1}^{\infty} 1/l^\nu$  as  $z \rightarrow 1$ , which is finite for  $\nu > 1$ . Show that this implies that for  $d > 2$  there is Bose-Einstein condensation for  $T > 0$ .
- (e) A theorist's rule of thumb says "For  $d \leq 2$ , long wavelength fluctuations prevent Bose-Einstein condensation". From your analysis in (d), explain why it can be said that the divergence of  $b_\nu(z)$  as  $z \rightarrow 1$  for  $\nu \leq 1$  prevents Bose-Einstein condensation. Illustrate the rule of thumb by writing down the integral for  $N$  at  $\mu = 0$  in  $\mathbf{k}$  space, and show that the contribution of modes near  $|\mathbf{k}| = 0$  is divergent for  $d \leq 2$ .