Emergent States of Matter

HOMEWORK SHEET 2

Due 5pm Mon 5 March, 2018 in the 569 ESM box.

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 2–1.
Consider the thermo-mechanical effect experiment sketched and described in lectures. Two vessels containing HeII at the same temperature and pressure are connected at the bottom by a superleak. The pressure in the left vessel is increased by ΔP, and this causes fluid to flow through the superleak into the right vessel, raising the level of the fluid there by an amount Δz.

(a) Explain in words, using the two-fluid model concept, what is happening. What is the *sign* of the temperature change in the left vessel? What is the *sign* of the temperature change in the right vessel? Explain what happens if the capillary diameter is not assumed to be infinitesimal.

(b) Starting from the thermodynamic relation $E - TS + PV = \mu N$ and using the first law of thermodynamics, or otherwise, show that in particle equilibrium

$$\frac{\Delta T}{\Delta P} = V/S.$$

Here, ΔT and ΔP are temperature and pressure differences between the two vessels, S is the total entropy of the system and V the total volume. State carefully and explicitly which properties of He II you invoked. In what sense is this an equilibrium situation?

Question 2–2.
Consider a cylindrical bucket of He II, with axis along the z-direction. A vortex of strength $n = 1$ is present at a distance r_v from the centre of the bucket. Calculate the angular momentum L_z of the superfluid, starting from the formula

$$L_z = \int_{bucket} r dr\, d\theta\, dz\, \rho_s r v(r).$$

Give careful consideration to the region of integration. Neglect the variation of ρ_s in the vortex core.

Question 2–3.
In this problem we will derive the formula for the interaction energy between two vortices of strength n_1 and n_2, separated by a distance d, near the centre of a rotating bucket of He II of radius $R \gg d$.

We consider two parallel vortices with vorticity n_1 and n_2 for generality with associated velocity fields $\mathbf{v}_1(\mathbf{r})$ and $\mathbf{v}_2(\mathbf{r})$ respectively. If only vortex 1 were present the energy would be E_1 and if only vortex 2 were present the energy would be v_2. The interaction energy E_{int} is defined as the energy of the system when both vortices are present minus $(E_1 + E_2)$.

(a) Write down the expression for E_{int} as an integral in terms of $\mathbf{v}_1(\mathbf{r})$ and $\mathbf{v}_2(\mathbf{r})$.

(b) Evaluate the integral carefully and show that

$$E_{int} = A \log(R/d)$$

and calculate the constant A; here d is the separation between the vortex cores and R is the radius of the bucket. You may assume that the vortices are situated near the centre of the bucket in making necessary approximations.

Question 2–4.

First sound—familiar sound waves in air, for example— is a density fluctuation in the particles that comprise a given system. HeII has two velocity fields $\mathbf{v}_n$ and $\mathbf{v}_s$, and two density variables $\rho_n(T)$ and $\rho_s(T)$ whose sum is the total mass density ρ of HeII. The total current density is given by the sum of the normal and superfluid currents: $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$. First sound is an in-phase motion: $\mathbf{v}_n = \mathbf{v}_s$, and leads to a propagating wave of density in the system.

In this question we are going to consider a propagating wave mode in which the normal and superfluid components move out of phase with each other, so that the total density does not change, and the total current is zero. This is a counter-flow wave, and since ρ does not change, $\rho_n(T)$ and $\rho_s(T)$ separately vary in a wavelike manner in space and time. The relative proportion of ρ_n and ρ_s is a measure of the temperature, so we can regard this sort of motion as a wave in temperature or entropy; it is not a wave in density or pressure of the real Helium, because the total density does not change. Since heat is only carried by the quasi-particles, a wave in heat corresponds to a wave in the density of the quasi-particles, i.e. ρ_n. This means that second sound can be regarded as a first sound wave in the gas of quasi-particles of the system!

In fact, any system, with quasi-particle excitations with a phonon-like dispersion relation, will exhibit second sound.

(a) Consider a gas of particles or quasi-particles with energy spectrum $\epsilon = pc_1 = c_1(\hbar k)$, where p is momentum and k is wavenumber. The particles are Bosons. Show that the equation of state is $P = E/3V$, where E is the internal energy and P is the pressure, V is the volume. This result should be familiar to you from the case of photons.

(b) Consider a compressible gas with density $\rho(\mathbf{r}, t)$ and equation of state for the pressure $P = P(\rho)$. Suppose $F(\mathbf{r}, t)$ is some property of the fluid that depends on space and time, such as the density, the temperature or even a component of the velocity vector $\mathbf{v}$. Using the chain rule of differential calculus, and the definition of the fluid velocity $\mathbf{v} = \dot{\mathbf{r}}$, show that the time-development of F following the fluid is given by the total derivative

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F$$

The meaning of this is that the partial derivative indicates derivatives at a fixed point in space, but the total derivative follows a point in the fluid as it moves.

(c) If we apply the above result to each component of the velocity $\mathbf{v}$, we get the expression for acceleration of an infinitesimal fluid element. By considering a fluid element moving under the influence of a local pressure gradient, show that Newton's second law of motion can be written as

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p$$

ignoring viscosity. [Hint: from the divergence theorem, you can derive the result $\int \nabla F dV = \int F d\mathbf{S}$, which you can use in conjunction with the definition of pressure as force per unit area to get "force = mass × acceleration".]

(d) For small velocities, show that the pressure and/or density in the gas satisfies a wave equation, with the speed of sound given by $c^2 = \partial P/\partial \rho$

(e) Hence show that the speed of first sound in the gas of photons, i.e. the speed of second sound in HeII c_2, is given by $c_2 = c_1/\sqrt{3}$.

(f) The result that you have just obtained is an asymptotic result as $T \to 0$, and can be obtained from a full treatment of the superfluid hydrodynamics equations, with more work. Where in the argument given above did we assume $T = 0$? Try to guess the form of the correction terms for $T > 0$.