

Emergent States of Matter

HOMEWORK SHEET 1

Due 5pm Tue 23 Feb 2021 online.

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 1–1.

Shortly after the discovery of the high temperature superconductors, known as the “cuprate superconductors”, the following facts were determined:

- superconductivity arises due to the pairing of electrons to form bosonic Cooper pairs
- the Cooper pairs may be loosely thought of as a sort of bound state (described and analyzed in more detail later in the course), but the characteristic size of this state is not thousands of Angstroms, as it is in most superconductors known at that time: instead the characteristic dimensions are of order a couple of Angstroms.
- in the normal state, the cuprates form *antiferromagnets* with a pattern of spins in a configuration that can be thought of as alternating spin-up, spin-down, spin-up, spin-down, ... as one moves through the crystal lattice.

Given the above facts, what orbital angular momentum state would you expect the Cooper pair electrons to be found in? Explain briefly your reasoning.

Question 1–2.

- The order parameter for superfluid helium is a complex number $\psi(\vec{r})$ defined at every point in three dimensional space \vec{r} . The core of a topological defect is the set of points where $\psi(\vec{r})$ vanishes. From this information, explain why it is that the topological defect must be a line-like object, i.e. a vortex.
- Consider the order parameter $\psi(\vec{r}) = \psi_0(\cos\theta(\vec{r}), \sin\theta(\vec{r}))$ and assume that the system is in a space of dimension $d = 2$. Let $\theta(\vec{r}) = \phi + \theta_0$ where the position vector $\vec{r} = (r, \phi)$ in polar coordinates. Sketch the order parameter field and the superfluid velocity field $\vec{v}(\vec{r}) \equiv \nabla\theta(\vec{r})$ corresponding to the situations where (i) $\theta_0 = 0$; (ii) $\theta_0 = \pi/2$; (iii) $\theta_0 = \pi$.
- A vortex with strength n greater than unity is described as in (b) above, but with $\theta(\vec{r}) = n\phi + \theta_0$. Sketch the order parameter field that corresponds to (i) $n = 2$; (ii) $n = -1$.
- Sketch the order parameter field for a pair of vortices in two dimensions, one vortex with $n = 1$ the other with $n = -1$.
- Topological defects are stable because it is not possible to make a small rearrangement of the direction of the order parameter within a finite region, and remove the defect. In more sophisticated language, we might say that there is no continuous deformation of the order parameter which will remove the defect configuration. Explain briefly, with a sketch or cartoon if you like, why it is impossible to remove a single vortex configuration without tampering with the orientation of the order parameter at an arbitrary distance from the core. Contrast this with the case of a “spin wave” configuration, described by $\phi(\vec{r}) = 0.01 \sin x$. Is a spin wave a topological defect?
- Is the configuration of a pair of vortices a topological defect?
- Consider an order parameter which is a n -component vector $\psi(\vec{r}) = (\psi_1(\vec{r}), \psi_2(\vec{r}), \dots, \psi_n(\vec{r}))$, and \vec{r} is a d -dimensional position vector as usual. Generalising (a) show that the dimension of the core of possible topological defects is $d_c \leq d - n$.
- Consider a system with $n = 3$ and $d = 2$. Start with a vortex configuration which is identical to that of a system with $n = 2$ and $d = 2$, and show that this vortex is not a stable topological defect when $n = 3$ and $d = 2$. You can do this by sketching a series of cartoons of the order parameter field, if you wish.

Question 1–3.

This question concerns the idea of a *functional* that we discussed in class. In the first parts, you will remind yourself about a simple one-particle problem in classical mechanics, and then move on via a many-body problem to a problem in continuum field theory. In the second, you will apply the same reasoning to the Landau free energy.

Consider a classical particle of mass m in one dimension subject to a potential $V(x)$. The Lagrangian $L \equiv T - V$ where the kinetic energy is $T = m\dot{x}^2/2$, and the action $S \equiv \int_{t_0}^{t_f} L dt$, where the initial and final times are denoted by t_0 and t_f . Note that the action is a functional of the trajectory $x(t)$, so we can write $S = S\{x(t)\}$.

- (a) Let's suppose that you don't know Newtonian mechanics, and you try to construct the equation of motion using a variational method. To do this, suppose that the actual path followed by the particle, and the one which satisfies the equation of motion (which you don't know at this stage) is denoted by $\bar{x}(t)$. Let $\eta(t)$ be some perturbation of the trajectory (ie. $\eta(t) = \delta x(t)$), so that $x(t) = \bar{x}(t) + \eta(t)$. Calculate the change in the action $\delta S \equiv S\{\bar{x}(t) + \eta(t)\} - S\{\bar{x}(t)\}$ to first order in $\eta(t)$. To do this, you will need to perform an integration by parts. Show that requiring that $\bar{x}(t)$ is an extremum of the action, so that $\delta S = 0 + O(\eta(t)^2)$ results in Newton's equation of motion. Give the differential equation satisfied by the Lagrangian L . You can obtain this by writing $\delta S = \int$ some mess $\delta x(t)$. The quantity $\delta S/\delta x(t)$ is called the functional derivative, and your calculation should have demonstrated that setting the above functional derivative equal to zero results in the Euler-Lagrange equation that you already know.
- (b) Now consider a system of N particles of mass m separated by springs of equilibrium length a with spring constant k . Denote the displacement from the equilibrium position by $\phi_i(t)$ where $i = 1 \dots N$ labels the particles. Assume periodic boundary conditions. Write down the Lagrangian and the action, and using the variational principle above, calculate the equation of motion for the i th particle and the equation satisfied by the Lagrangian.
- (c) Now consider an elastic string of mass per unit length ρ and string tension T , held between two supports a distance X apart, and denote the displacement by $\phi(x, t)$. Note that ϕ is a function of horizontal position x , and that x plays the same role as the label i in part (b). Write down the Lagrangian density \mathcal{L} , in terms of which the Lagrangian is $L = \int_0^X \mathcal{L} dx$. To do this, I recommend that you assume that the displacements are small with respect to the equilibrium separation a , so that ϕ_i is replaced by $\phi(x)$ at $x = ia$, $\phi_{i+1} - \phi_i = a\partial_x\phi(x)$ evaluated at $x = ia$. In this continuum limit, where $a \rightarrow 0$, $N \rightarrow \infty$, $aN = X$, the summation $\sum_{i=1}^N$ is replaced by an integral:

$$\sum_{i=1}^N \rightarrow \frac{1}{a} \int_0^X dx$$

Your answer for \mathcal{L} should be in terms of $\partial_t\phi(x, t)$ and $\partial_x\phi(x, t)$.

- (d) Let's find the equation of motion satisfied by the field $\phi(x, t)$, using the variational principle. This time, we denote the configuration in space-time of the field $\phi(x, t)$ which satisfies the equation of motion by $\bar{\phi}(x, t)$. We suppose that there is a perturbation of this trajectory $\delta\phi = \eta(x, t)$, and a corresponding change to the action δS . Note that the trajectory is actually a sheet in space-time, because at any instant in time t^* , you have a configuration $\phi(x, t^*)$, which is a one-dimensional curve as a function of x . If you now vary the time t^* , the curve sweeps out a two-dimensional sheet or manifold. Calculate δS to first order in $\eta(x, t)$, and by requiring that the action be stationary with respect to perturbations, show that the equation governing the trajectory $\bar{\phi}(x, t)$ is the wave equation.
- (e) Consider the Landau free energy $L \equiv \int \mathcal{L} dx$ as a functional of the order parameter $\phi(x)$

$$L = \int \left[\frac{\gamma}{2} (\partial_x \phi(x))^2 + \frac{at}{2} \phi(x)^2 + \frac{b}{4} \phi(x)^4 \right] d^d x.$$

Here, as discussed in class, γ , a and b are positive constants of order unity and $t \equiv (T - T_c)/T_c$ is the relative temperature. In thermal equilibrium, the Landau free energy functional is minimised. What is the equation satisfied by the order parameter $\phi(x)$? What is the equation satisfied by the Landau free energy density \mathcal{L} ?