Analogue of Hawking Radiation in a Bose-Einstein Condensate

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Abstract

This paper will explore the possibility of observing the analogue of a black hole and Hawking radiation in a Bose-Eistein condensate.

1 Introduction

Einstein's general theory of relativity models gravity as the curvature of 4-dimensional "spacetime" by the mass and energy living in the spacetime. The distortion caused by a ball sitting in the center of a rubber sheet is a way of imagining how this mechanism works. If no ball is present, then the 2-dimensional rubber sheet is no longer distorted.

If the source of matter is dense enough, then a black hole forms, a region of extreme spacetime curvature, from which nothing can escape, not even light. For example, for an object the mass of the sun, its radius needs to be around 3 kilometers in order to be a black hole.

Classically, a black hole is black. It does not emit anything. However, when general relativity is combined with quantum field theory, a surprising result arises. Namely, black holes do in fact radiate. This type of radiation is known as Hawking radiation, in honor of Stephen Hawking who formulated this theory. Although a black hole and Hawking radiation have not been detected experimentally, there has been increasing interest in observing analogue of these phenomena in several condensed matter systems, with Bose-Einstein condensates (BEC) being the most promising.

2 Hawking Radiation

In this section, a description of the Hawking radiation process is given which follows the discussion in [5]. For simplicity, assume that the metric is given by the spherically symmetric Schwarzschild geometry (this describes a non-rotating, electrically neutral black hole). Therefore the line element is

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1}$$

where the speed of light has been set to one. The mass of the black hole is M, and the horizon is located at $r = 2M$.

In quantum field theory the spacetime is filled with electromagnetic fluctuations that produce pairs of photons. These pairs come in and out of existence in a time $\Delta t = \hbar/\Delta E$, where ΔE is the amount of energy borrowed to produce such a pair. If a pair of such photons is produced near the horizon, one of energy $-E$ and the other of energy E, then it is possible for the negative energy photon to cross the horizon within a time $\hbar/E$ and not recombine with the positive energy photon. An observer inside the horizon measures the energy of a nearby photon to be:

$$E = -\left(\frac{2M}{r} - 1\right)^{-1/2} p^r, \qquad (r < 2M) \tag{2}$$

where p^r is the radial component of the photon's four-momentum. Note that p^r is negative for an ingoing photon (therefore the energy measured is positive). Meanwhile the other photon is allowed to escape to infinity. The energy of a photon at infinity as measured by an observer freely falling toward the horizon is given by:

$$E = \frac{h}{8\pi M}. \tag{3}$$

Hawking's more rigorous calculation showed the typical photon energy to be

$$E = \frac{\hbar}{8\pi M}. \tag{4}$$

Hawking also showed that the spectrum is characteristic of a black body with a temperature

$$T = \frac{\hbar}{8\pi k_B M}. \tag{5}$$

The characteristic photon wavelength is

$$\lambda = \frac{hc}{E} = 2\pi^{3/3} c\sqrt{A}, \tag{6}$$

where $A = 16\pi M$ is the area of a Schwarzschild black hole. This result will come up again when looking at the analogues of Hawking radiation in a Bose-Einstein condensate.

Since the black hole is losing energy (or equivalently mass) by the absorption of negative energy photons, the black hole acquires a finite lifetime. The lifetime of a black hole can be shown to be proportional to M^3. A black hole whose mass is 10^{10} kg has a lifetime equal to the age of the universe. On the other hand a black hole of 10^6 kg evaporates in one second, with most of the radiation emitted as gamma rays. Such black holes could have been formed in the early universe, but such events have yet to be observed.

3 Bose-Einstein Condensates

The complex wave function of the condensate obeys the nonlinear Gross-Pitaevskii equation:

$$i\hbar\partial_t\psi(\vec{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x},t) + \lambda|\psi|^2\psi. \tag{7}$$

The external potential term has been omitted for simplicity. Let ψ be written in the following way:

$$\psi = \sqrt{\rho}\exp(-i\theta m/\hbar). \tag{8}$$

Plugging in eq. 8 into eq. 7 and taking the real and imaginary parts, one arrives at the following two equations [2]:

$$\partial_t\rho + \nabla\cdot(\rho\nabla\theta) = 0. \tag{9}$$

$$\frac{\partial}{\partial t}\theta + \frac{1}{2}(\nabla\theta)^2 + \frac{\lambda\rho}{m} - \frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = 0. \tag{10}$$

These equations are the usual hydrodynamic equations for a fluid of density ρ. The first equation is the equation of continuity for the fluid density ρ whose flow velocity is given by $\vec{v} = \nabla\theta$. In other words, the flow is irrotational. The second equation is the

Bernoulli equation plus an extra "self-interaction" term, $V_q = -\frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$. From eq. 10, one can identify the enthalphy, the equation of state, and the speed of sound [2]. The specific enthalpy is given by:

$$h = \int_0^p \frac{dp}{\rho(p)} = \frac{\lambda\rho}{m} \tag{11}$$

The equation of state of this fluid is

$$p = \frac{\lambda\rho^2}{2m} \tag{12}$$

and the speed of sound is equal to:

$$c_s^2 = \frac{\partial p}{\partial \rho} = \frac{\lambda\rho}{m}. \tag{13}$$

The fluid is thus barotropic since the equation of state is a function of density only [2].

Following [3], write the p, ρ, and θ as:

$$p(\vec{x},t) = p_0(\vec{x},t) + p_1(\vec{x},t) \tag{14}$$

$$\rho(\vec{x},t) = \rho_0(\vec{x},t) + \rho_1(\vec{x},t) \tag{15}$$

$$\theta(\vec{x},t) = \theta_0(\vec{x},t) + \theta_1(\vec{x},t). \tag{16}$$

The subscript 0 represents a background which solves equations eqs. 3, 4, and 6, where the fourth term in eq. 4 has been assumed to be negligible (this corresponds to the low-momentum limit) [2]. The subscript 1 represents fluctuations about the background. The goal is to find the linearized equations of motion for the fluctuations. Plugging in eqs. 8, 9, 10 into 3, 4, and 6 and keeping terms linear in the fluctuations, one obtains [3]:

$$\frac{\partial \rho_1}{\partial t} + \nabla\cdot(\rho_1\nabla\theta_0 + \rho_0\nabla\theta_1) = 0 \tag{17}$$

$$\rho_0(\frac{\partial\theta_1}{\partial t} + \nabla\theta_0\cdot\nabla\theta_1) = p_1 \tag{18}$$

$$p_1 = c_s^2\rho_1. \tag{19}$$

These equations can be combined to a second order differential equation for θ_1. The resulting equation is equal to:

$$\frac{\partial}{\partial t}\left(c_s^{-2}\rho_0\left(\frac{\theta_1}{\partial t} + \vec{v_0}\cdot\nabla\theta_1\right)\right) = \nabla\cdot\left(\rho_0\nabla\theta_1 - c_s^{-2}\rho_0\vec{v_0}\left(\frac{\partial\psi}{\partial t} + \vec{v_0}\cdot\nabla\theta_1\right)\right). \tag{20}$$

If we write the following matrix:

$$g^{\mu\nu} = \frac{1}{\rho_0 c_s}\begin{pmatrix} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c_s^2\delta^{ij} - v_0^i v_0^j) \end{pmatrix} \tag{21}$$

and define $g = [\det(g^{\mu\nu})]^{-1}$, then eq. (20) can be written in the following "relativistic," Lorentz invariant form [3]:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\psi_1\right) = 0 \tag{22}$$

The interpretation of eq. 16 is that phonons obey a massless scalar field in a curved spacetime with a metric of the form:

$$g_{\mu\nu} = \frac{1}{\rho_0 c_s}\begin{pmatrix} -(c_s^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta^{ij} \end{pmatrix}. \tag{23}$$

The metric allows the formation of horizons if the local sound speed is equal to the velocity of the background flow. In that case, $c_s^2 - v_0^2$ (the term proportional to g_{00}) vanishes, which is the condition for the formation of a horizon in black hole physics. For example, in the Schwarzschild metric, $g_{00} = 0$ says that the horizon radius is equal to twice the mass of the black hole. Sound waves inside the horizon can no longer cross this horizon since the sound speed of the wave is less than that of the background flow ($c_s^2 - v_0^2 < 0$). This analysis is valid for wavelengths $\lambda >> \frac{2\pi\hbar}{mc_s}$. Wavelengths shorter than this lie outside the hydrodynamic limit and break the Lorentz invariance. [4].

4 Experimental Setup

4.1 Laval Nozzle

In [1, 2], two Laval nozzles are used to create a horizon and Hawking radiation. Figure 1 demonstrates the experimental set-up.[1] In such case, the region in the middle is supersonic and it is surrounded by a subsonic region on either side. The equations that follow in this section have been taken from [1, 2].

The equations of motion are in this case one-dimensional, depending on z, the distance along the symmetry axis of the set-up. The velocities transverse to this direction are negligible. The continuity equation then takes on the form

$$\rho(z)A(z)v(z) = J = constant, \tag{24}$$

where $A(z)$ is the cross sectional area. Euler's equation becomes

$$\rho v\frac{dv}{dz} = -\frac{dp}{dz}. \tag{25}$$

Defining $X' = dX/dz$, one obtains from from the continuity equation

$$\rho' = -\rho\frac{(Av)'}{(Av)} = -\rho\left[\frac{A'}{A} + \frac{v'}{v}\right] = -\rho\left[\frac{A'}{A} + \frac{a}{v^2}\right], \tag{26}$$

[1] figure taken from from [2]

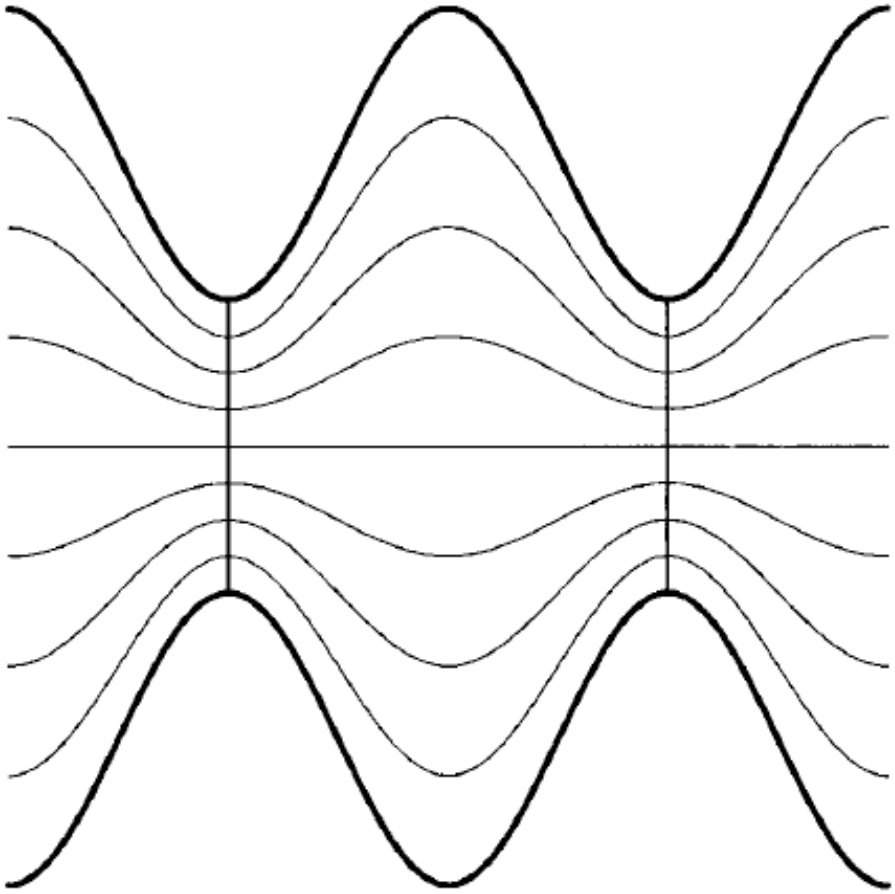

Figure 1: Two Laval nozzles enclosing a region of supersonic flow.

where a is the fluid's acceleration. Euler's equation yields

$$\rho a = -\frac{dp}{d\rho}\rho'. \tag{27}$$

Combining equations 19, 20 gives the "nozzle equation"

$$a = -\frac{v^2 c^2}{c^2 - v^2}\left[\frac{A'}{A}\right]. \tag{28}$$

At the acoustic horizon, $c^2 - v^2$ goes to zero. Therefore A' needs to go to zero so that the acceleration is finite there. The acoustic horizon thus form at the part where the nozzle is narrowest. The form of the acceleration at the horizon is:

$$a_H^2 = \left.\frac{c^4 A''/A}{2 + \rho(d^2p/d\rho^2)/c^2}\right|_H. \tag{29}$$

The acceleration of the fluid at the horizon is therefore determined by the speed of sound, the geometry of the horizon, and the equation of state. Note that the acceleration at

the horizon can have both signs. If it is positive, then this is a black hole horizon. On the other hand, deceleration corresponds to a "white hole" horizon.

In talking about Hawking radiation, one is interested in the limit of the the quantity

$$g = -\frac{1}{2}\frac{d(c^2 - v^2)}{z} \tag{30}$$

evaluated at the horizon. This quantity is known as the surface gravity. For this system, the surface gravity has the form:

$$g_H = \pm \frac{c_H^2}{\sqrt{2A_H}}\sqrt{1 + \frac{\rho}{2c^2}\frac{d^2p}{d\rho^2}}\Bigg|_H \sqrt{A_H^{''}}. \tag{31}$$

For a BEC, g_H has the following simpler form:

$$g_H = \pm\frac{c_H^2}{\sqrt{A_H}}\sqrt{3A_H^{''}/4}. \tag{32}$$

The horizon acts as a black body emitting radiation at a temperature

$$k_B T_H = \frac{\hbar g_H}{2\pi c_H} = \hbar\frac{c_H}{2\pi\sqrt{A_H}}\sqrt{\frac{3A_H^{''}}{4}}. \tag{33}$$

The radiation spectrum peaks at

$$\omega_{peak} = \frac{c_H}{2\pi\sqrt{A_H}}\sqrt{\frac{3A_H^{''}}{4}}. \tag{34}$$

The corresponding wavelength is then:

$$\lambda_{peak} = 4\pi^2\sqrt{A_H}\sqrt{\frac{4}{3A_H^{''}}} \tag{35}$$

In Hawking radiation, the peak wavelength is on the order of the diameter of the hole. Here the same holds true since λ_{peak} is proportional to $\sqrt{A_H}$.

[1, 2] point out that for a BEC with a sound speed of 6 mm/s in a nozzle with $A_H^{''} \approx 1$ and a nozzle diameter of 1 micron, the Hawking temperature is expected to be around T_H=7 nK. However, they also point out that in recent experiments, researchers have been able to control the scattering length of the condensate, increasing it by up to a factor of 100. This means that the sound speed is augmented by a factor of 10. Therefore the Hawking temperature is around 70 nK, which is close to the condensation temperature of 90 nK.

5 Stability of the Black Hole

In [4], stabilities were checked for a BEC constrained to move in a one-dimensional ring-shaped external potential. The black hole solution obtained for such a system was

then perturbed by a sum of discrete modes. Their calculations showed that a sufficiently high frequency cutoff had little effect on their solution. Real negative frequency modes were present, meaning that the black hole solution is energetically unstable and therefore it dissipates. However, the time scale for such a process can be made long enough for a sufficiently cold, weakly interacting condensate.

More problematic instabilities are those with modes having complex frequencies. For high-enough cutoffs, however, the imaginary parts of these frequencies remained small relative to the real parts and independent of the cutoff frequency. These "dynamical" instabilities depend sensitively on three parameters, (U, b, w), introduced in this model. U is the external potential. w is an integer "winding number." $b \in [0, 1]$ is a number parameterizing the condensate density. Figure 2 shows a contour plot of the maximum of the absolute value of the imaginary part of all of the complex eigenfrequencies for $w = 7$.[2] The stable regions are the dark gray areas. The plot also shows a path (denoted by the upside-down L) which takes the system from a subsonic state to a supersonic state with the formation of a black hole. This path is minimally exposed to the instability regions (the white curves). Once a black has been formed, one can then move it to an unstable region where the black hole is destroyed in an explosion of positive- and negative-energy phonons which are created near the horizon, in similar fashion to the Hawking radiation process for a real black hole.

6 Conclusion

The Gross-Pitaevskii equation governs the behavior of a weakly interacting condensate. The behavior of low momentum modes behave like sound waves with a finite sound speed. When considering these modes only, the Gross-Pitaevskii equation has the form of a continuity equation plus Euler's equation describing the behavior of an irrotational fluid. The literature points out that these hydrodynamic equation are equivalent to a scalar field obeying the wave equation in curved spacetime. Thus one can create the equivalent of a black hole in a fluid having regions on which the background flow velocity is higher than the speed of sound. Such a sonic black hole can be observed to emit phonons, in analogy to the Hawking radiation process of gravitational black holes. Experiments in the near future should be able to demonstrate these results and thereby observe analogies of general relativistic effects in the lab.

References

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[2] figure taken from [4]

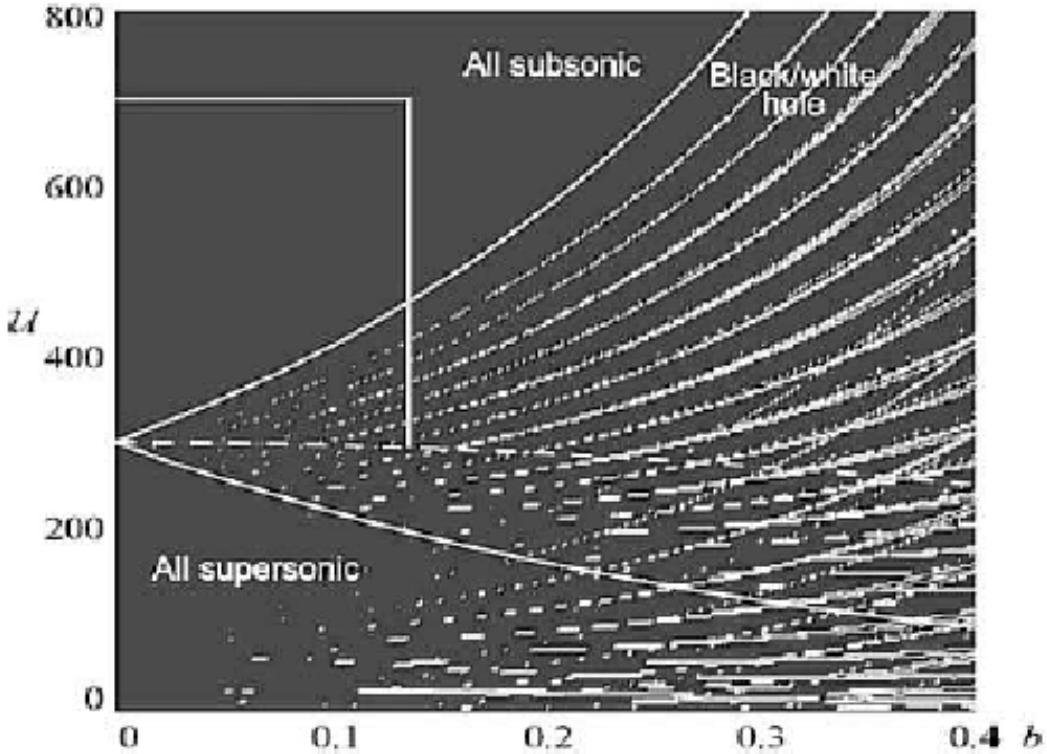

Figure 2: Maximum absolute value of the imaginary part of the eigenfrequencies. The vertical axis represents U, and the horizontal axis represents the parameter b.

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