Emergence of Anyons with Novel Statistics in 2-dimensional Systems

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Abstract

In certain two-dimensional systems, due to the interactions between electrons or atoms, a neither fermionic nor bosonic kind of quasi-particles can emerge. They are called anyons. In this essay, I will review the Abelian and non-Abelian statistics of anyons, and explain the quantum Hall effect which offers possible systems for anyons to exist. Experimental techniques to detect anyons and some evidence will also be mentioned. The identification of non-Abelian anyonic states will be a crucial step in topological quantum computation.

1 Introduction to the history and current status:

Anyons were theoretically predicted by J. M. Leinaas and J. Myrheim in 1977[1] and independently studied by F. Wilczek in 1982 [2] who gave these particles the name. As a new kind of quasi-particle excitations which go beyond the fermion-boson dichotomy, the theoretical construct of anyons is exciting enough. Later on, it was realized that by making use of the exchange statistics of non-Abelian anyons, topological quantum computation is possible. To live up to its name, the topological computation transforms the state of the system in a way that the result only depends on the topological class of the trajectories of those anyons. Small errors of the trajectories don't affect the the topological class, which makes the topological quantum computation fault-tolorent. In 2005, Sarma, S. D., Freedman, M. and Nayak, C.[3] proposed a design which can produce a topologically protected qubit on which a logical NOT operation is able to be performed.

Anyons are not just theoretical constructs. They are manifested in fractional quantum Hall effect, which is the first and so far the only system where the existence of Abelian anyons is convincing and this effect is described by Lauglin model. In 1983 [4], Laughlin provided a trial ground state wavefunction of a two-dimensional electron gas placed in the magnetic field to explain the $\nu = 1/3$ fractional quantum Hall effect observed in experiment. Laughlin's wavefunction also successfully predicts the existence of other $\nu = 1/n$ states and that the corresponding quasi-particles should have fractional electric charge e/n. Then it was realized by Halperin [5] and Arovas, Schrieffer and Wilczek [6] in 1984 that the fractional charge of the quasi-particles in the fractional quantum Hall effect also implies these excitations must obey fractional statistics, which makes the fractional quantum Hall effect a potential experimental system to find, and even manipulate anyons.

In a key development in the detection of anyons, in 2005, Vladimir J. Goldman, Fernando E. Camino, and Wei Zhou [7] reported that they directly observed the fractional statistics of Abelian anyons by using a fractional quantum Hall interferometer, where quasiparticles of the $\nu = 1/3$ fractional quantum Hall state encircle an island of the $\nu = 2/5$ state and thus a statistical phase is accumulated, which can be observed from the shift of interference fringes. But some other researchers pointed out that their results could be the product of phenomena not involving anyons. It should also be noted that there is no direct experimental evidence for the non-Abelian nature of the quasiparticles.

2 Anyonic statistics:

One basic principle of quantum mechanics is that the wavefunctions should satisfy certain symmetry properties under the exchange of identical particles. We will introduce now what symmetries can it be in 3+1-dimensional and 2+1-dimensional spacetime respectively.

In three-dimensional space, any loop one particle takes to move around another can be topologically deformed into a point without cutting through another particle, so it is equivalent to none of the particles moves at all [8]. Also, wrapping one particle all the way around the other is equivalent to the case that the particles are interchanged twice. So under a single interchange, the only two possibilities are for the wavefunction to change by a $\pm$ sign, corresponding to bosons and fermions. Things are different in two-dimensional space because winding a particle around another is not equivalent to none of the particles move at all, since the path can't be deformed into a point without cutting through another particle. So when interchange the particles twice both in the clockwise/ counterclockwise direction, the system will not necessarily come back to the original state, say a factor of $e^{2i\theta}$ may appear. And the clockwise and counter-clockwise paths are also topologically different because we can't continuously deform the clockwise path into counter-clockwise path without having the particles collide somewhere.

Thus, in two dimensions, interchange two particles will result in an arbitrary phase:

$$\psi(\mathbf{r_1}, \mathbf{r_2}) \rightarrow \mathbf{e^{i\theta}}\psi(\mathbf{r_1}, \mathbf{r_2})$$

Particles with any other values of θ between 0 and π are called anyons. But we should note that one class of anyons can only own a single value of θ. The overall phase of a quantum state matters, because even though multiplying a wavefunction by a phase doesn't affect the measured properties of this single wave, it can affect how this wave interferes with other waves.

3 Braid group and classification of anyons:

The braid group B_N contains elements each is a topologically equivalent class of trajectories in 2+1d that takes the particles of the system from their initial positions $R_1, R_2, ..., R_N$ to final positions $R_1, R_2, ..., R_N$[8]. Imagine there is a thread connecting the initial spatial and time position of each particle with itself in the final position, then a way to bring all these particles of the system to the final states is like braiding a group of threads. B_N describes all the possible ways to braid a given row of thread together.

We use the operators $\sigma_i/{\sigma_i}^{-1}$ to represent a counter-clockwise/clockwise exchange of particle i and $i+1$. The simplest case is that σ_i is one dimensional, so the wavefunction simply acquires $e^{i\theta}$ when we exchange two particles. If we first exchange the particles A and B, then exchange B and C, the state of the system will acquire a factor of $e^{i(\theta_1+\theta_2)}$ in total. Suppose instead B and C are exchanged first, obviously the wavefunction is multiplied by the same factor $e^{i(\theta_2+\theta_1)}$ as before. Apparently the order in which the particles are swapped doesn't make a difference. The particles are thus called Abelian anyons.

If there are g degenerate states (represented by orthonormal states $\psi_1, ..., \psi_g$) that all describe the particles being in positions $R_1, R_2, ..., R_N$, an interchange of two quasiparticles does not necessarily merely multiply the groundstate wavefunction by a phase factor, shifting the system to a different ground state is also reasonable. This means we need higher dimensional representation of σ_i, which should be a $g \times g$ unitary matrix $\rho(\sigma_i)$. The result of multiplying two matrices depends on the order in which they are multiplied. If $\rho(\sigma_1)$ and $\rho(\sigma_2)$ commute, these particles still obey Abelian braiding statistics. But in most cases two matrices don't commute, then the order in which the particles are switched is important. These particles are called non-Abelian anyons.

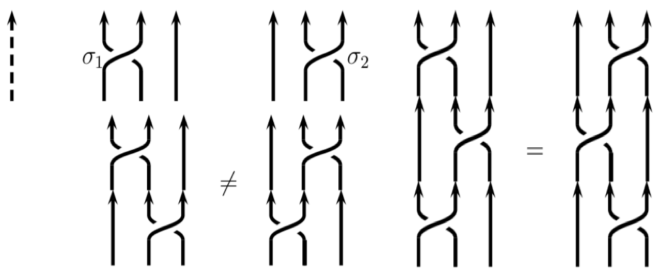


Figure 1: Non-Abelian braid operators [8]. Left top: basic exchange operators. Left down: the non-communitive property of non-Abelian anyons. Right: two different series of operations lead to the same final configurations.

Non-Abelian anyons:

The statistics of non-Abelian anyons is extremely interesting, which also implies their potential utility in topological quantum computation. While at the same time, the requirements for non-Abelian statistics to take place is harsher than Abelian one.

We consider a 3d fluid consisting a large number of identical particles. The collective behavior of these particles creates vortices that are localized at R_j. The wavefunctions for the ground states are $|\psi_\alpha(R_j)\rangle$. Different α stands for different members belonging to the set of the degenerate ground states.

Four defining characteristics of the non-Abelian states are summed up in this paper [9]. First is that there must be an energy gap between the ground states and the excited states. This guarantees that the system stays at the ground states under adiabatic changes. Second, the system must have a set of degenerate ground states, and the degeneracy should be exponentially large in the number of the vortices. Third is that the degeneracy should be robust. Fourth, when interchange two vortices, the system should transform from one ground state to another, and the transformation should only depend on the topology of the trajectory instead of the geometry. The degeneracy of the ground states and the different initial and final states guarantee the non-commutative property of the operation (interchange two quasiparticles) and thus the non-Abelian statistics.

4 Integer and fractional qauntum Hall effect:

Two-dimensional electronic systems will show Hall effect when subject to a perpendicular magnetic field. The Hall conductance is $\sigma_{xy} = \nu\frac{e^2}{h}$. The filling factor $\nu = n\frac{\Phi_0}{B}$, where n is the density of electrons.. At low temperature and strong magnetic field, quantum Hall effects will be formed, which is characterized by ν can take on either integer or factional values.

The integer quantum Hall effect, for which ν is an integer, can be understood from Landau quantization. In 2 dimensions, when non-interacting charged particles subjected to a magnetic field, their orbits are quantized. The energy levels of these orbits, the Landau levels, are the energy levels of quantum oscillators: $En = \hbar\omega_c(n + \frac{1}{2})$, where $\omega_c = \frac{eB}{m}$. The degeneracy of each Landau level is $N = \frac{BA}{\Phi_0}$. Thus when the fermi level lies between the ν^{th} and $\nu + 1^{th}$ Landau levels, the electron density of the system is $n = \nu * N/A = \nu\frac{B}{\Phi_0}$. From this we can get $\nu = n\frac{\Phi_0}{B}$, which is exactly the ν^{th} plateau of the Hall conductance observed in experiment. So the integer quantum Hall effect is the result of the existence of Landau levels in a non-interacting charged particle system subjected to perpendicular magaetic field.

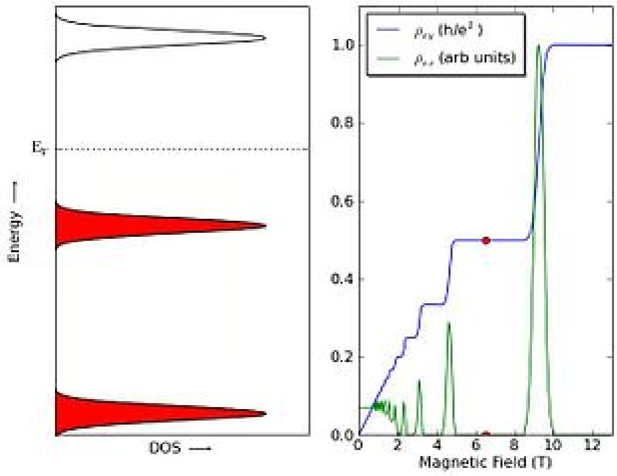


Figure 2: The connection between the integer quantum Hall effect and Landau levels

As a contrast, the fractional quantum Hall effect relies on the electron-electron interaction. However, people found that this can be mapped to a system composed of composite fermions [10], between which the interactions are negligible to a good approximation. Then these charged composite fermions can be treated the same way as the non-interacting electrons in the integer quantum Hall effect. In another word, the factional quantum Hall effect is the integer quantum Hall effect of composite fermions. Composite fermions are formed when electrons try to minimize the interaction by capturing quantized votices, which leads to the fact that the composite fermions feel a much smaller magnetic field than electrons:

$$B^* = B - 2pn\Phi_0$$

where $2p$ is the number of vortices bound to an electron to form a composite fermion. Thus the filling factor for the composite fermions is $\nu^* = n\frac{\Phi_0}{|B^*|}$. So,

$$\frac{n\Phi_0}{\nu^*} = \frac{n\Phi_0}{\nu} - 2pn\Phi_0$$

$$\Rightarrow \nu = \frac{\nu^*}{2p\nu^* \pm 1}$$

Now we can see since the composite fermion filling factor ν^* is an integer, the electron filling factor ν is factional. And for the charge e^* on the quasiparticle excitations of this state, we have:

$$e^* = \pm \frac{e}{2p\nu^* + 1}$$

Other than the composite fermion theory, Laughlin wavefunction also successfully explains the fractional quantum Hall effect.

5 Exerimental evidence of the existence of Anyons:

As I mentioned in Part 1, it was realized by Halperin [5] and Arovas, Schrieffer and Wilczek [6] that the fractional charge of the quasi-particles in the fractional quantum Hall effect is closely related to the fractional statistics, which makes FQHE the most promising system for the realization of anyons. Since Abelian anyons are defined by the phase that they acquire when they travel all the way around another, it is reasonable to design an experiment where the phase difference could be detected, which naturaly leads to interferometry.

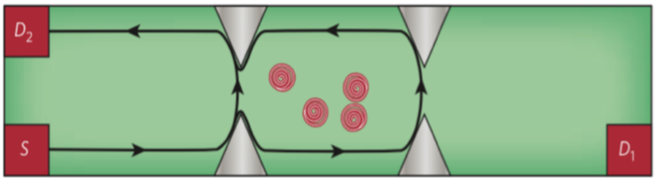


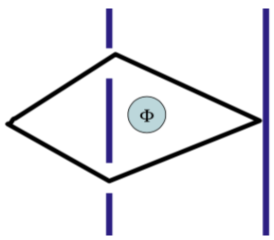
Figure 3: Fabry-Perot interferometer

Figure 3 [9] is the structure of a Fabry-Perot interferometer, which is a powerful technique for detecting anyonic properties. The two black lines stand for the two possible trajectories of the current. The current transports along the edge of the Hall bar and travel to the other side under the tunneling effect at two constrictions, which serve as beam spliters. The red circles are localized quasiparticles trapped in the loop. If we assume the tunneling amplitude at 1st and 2nd constrictions are t_1 and t_2 respectively [11], then the conductance of the interferometer is

$$G \propto |t_1|^2 + |t_2|^2 + 2Re\{t_1^* t_2 * e^{i\phi}\}$$

where ϕ is the sum of the phase due to both the Aharonov-Bohm effect (see below) and the exchange of quasiparticles. What we expect is that we can observe the phase changes ($\Delta\phi$)

discretely in the experiment, which should by 2θ due to the quasiparticle enter or exit the interferometer.



Aharonov–Bohm effect [12]: In a system with non-interacting electrons subjected to a strong magnetic field. The relative phase between two interference paths will be $2\pi\frac{\Phi}{\Phi_0}$, where Φ is the magnetic flux enclosed by the interference loop.

Recent studies have shown encouraging, even thought not definite, results. In [11], the researchers measured the diagonal resistance R_D as a the voltage bias on the plunger gate V_P is swept at a steady rate. The voltage is to change the encircled area and will result in oscillations of R_D. Figure 4 shows the result at $\nu = 7/3$ quantum Hall state. The black line is the best fit of data below $-20mV$. The blue curve is the black one shifted by $2\pi/3$. And the light blue curve is further shift the phase by $2\pi/3$. This is associated with an anyon randomly entering or exiting the interferometer and demonstrate the Abelian anyonic braiding statistics of the $e/3$ anyons for the $7/3$ fractional quantum Hall state.

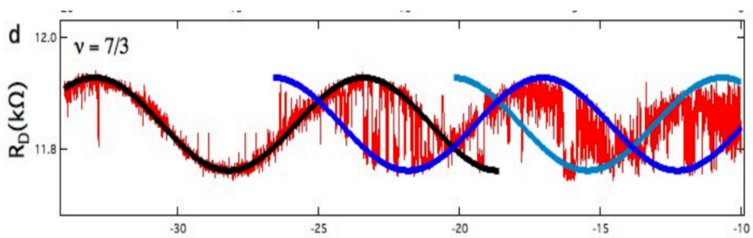


Figure 4: The interference measurement at $\nu = 7/3$ quantum Hall state

The first quantum Hall state suspected of being non-Abelian is the $\nu = 5/2$ state. Even though several experiments have been proposed to probe non-Abelian states, there is no direct experimental evidence for the non-Abelian nature of the quasiparticles. For the $\nu = 5/2$ Moore–Read state, no interference takes place when a non-Abelian $e/4$ quasiparticle interferes around an interference loop which encircles an odd number of similar quasiparticles [9]. On the contrast, interference would take place when that number is even. In this case one of two interference patterns will be seen, between which the phase is π, and the choice is determined by the ground state that the localized quasiparticles are in. Recent studies show that as the voltage on the side gate changes, the interferometer's resistance oscillates, and the periodicity of the oscillations switches between two values [9].

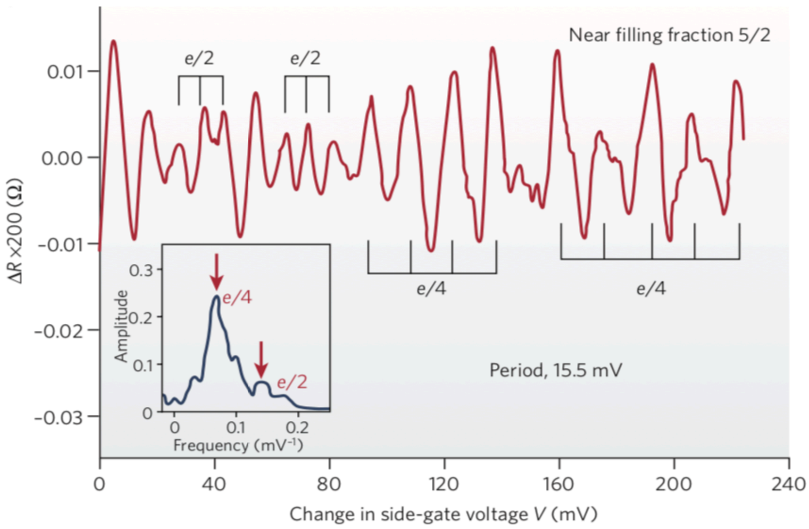


Figure 5: The switching of periodicities from the $\nu = 5/2$ Fabry–Pérot interferometer [13]

6 Quantum computation with Non-Abelian Anyons:

To build a topological quantum computer, one needs Non-Abelian anyons.

Place Anyon pairs along a line and then move the adjacent anyons around one another in a designed order, which produces a braiding of all the world lines of anyons [14]. The calculation of a topological quantum computer can be represented as a set of braids in spacetime. The result is a matrix combined from the matrices corresponding to each manipulation.

Can the topological quantum computer perform any quantum computations the traditional quantum computer can do? The answer is yes. Any quantum circuit can be simulated to an arbitrary degree of accuracy using a combination of a so-called controlled NOT (or CNOT) gate and single qubit rotations. It was showed by a team in 2005 how to construct CNOT gate to an accuracy of two parts in 10^3 by braiding six anyons [14].

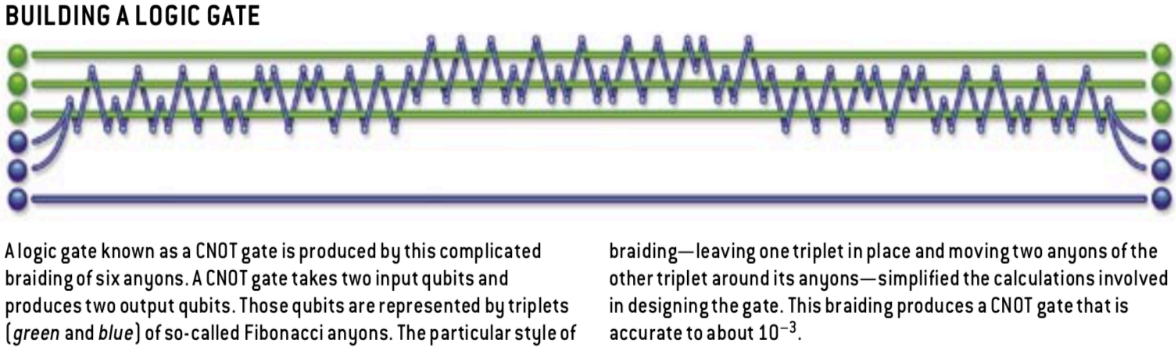


Figure 6: Building the CNOT gate [14]

Making practical quantum computers requires a lower error rate. Since anyons are just like fermions, they cannot occupy the same state, the world lines of two anyons cannot intersect or merge. This ensures that the topological property of the braided threads can't be changed by small perturbations, thus may provide an error-resistent approach to quantum computing. And different from the traditional quantum computation, the topological quantum computation is approximate, but actually it can reach any given accuracy. We only need to increase the number of number of twists to get a finer accuracy. It is fortunate that the required number of twists increases very slowly as the accuracy increases, so high accuracy is not just theoretically possible.

7 Conclusion:

Anyons are important as well as fascinating in more than one aspects, but the first is it breaks the previous rule that the wavefunction are either symmetric or antisymmetric under the exchange of identical particles, thus leading us to another part of quantum mechanics. I have reviewed in this paper how the notion of quasiparticle excitations helps with the understanding of fractional quantum Hall effect, and how the factional quantum Hall effect in the other way around offers us a possible theoretical and experimental platform to look for anyons. I didn't cover the part about how the candidate systems accomodating anyons are proposed, which is a regret. But I did review the detection techniques and some encouraging results. As we can see, the physics about anyons haven't been fully understood and awaiting for future research.

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