# Emergent Electrodynamics in a Skyrmion Lattice

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Abstract: When an electron travels through a non-uniform magnetic structure, it constantly experiences forces which attempt to orient it with the local magnetization. An emergent electrodynamics can be used to describe the net effect of these re-orientations. This essay focuses on Skyrmion lattices, topological quantum numbers and how topological properties of our lattice is inherited by the emergent fields. There is a discussion on the Quantum Hall Effect to motivate topological quantum numbers and the experiment also uses the Hall Effect as a means of data collection.

# 1 Introduction

An astonishing yet convenient aspect of physics is the abundance of analogies. For instance, an inductor in series with a charged capacitor acts as a mass on a spring, or, better yet, an electromagnetic field can be thought of as a continuous system of harmonic oscillators. With analogies we clarify the physics of abstract phenomena with a more suitable visualization for our minds eye. These sorts of comparisons are very important to effectively "visualize" modern day physics problems.

Advances in material sciences has provided physicists with many forms of exotic matter, which, in turn, reveals new phenomena. For instance, when electrons traverse non-collinear magnetic structures [1] the spin of the electron has to continuously reorient itself in an attempt to align its own magnetic moment with the ambient field (internal + external)[1]. The dynamics of the electron can be described nicely as a fictitious electrodynamics. This emergent electrodynamics is essentially an intuitive mechanism, like the previous examples, and allows accumulations of Berry phases to be thought of as a sort of Aharonov-Bohm phase.

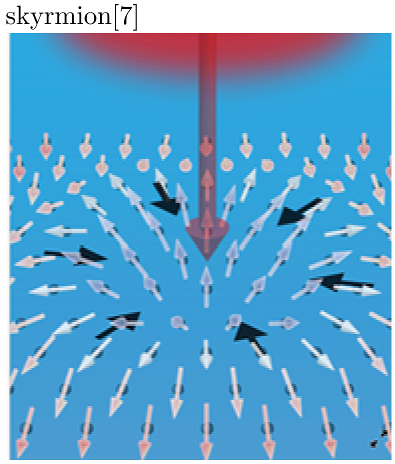
This essay introduces some key aspects of this emergent electrodynamics as well as an introduction to topological quantum numbers, and their importance. Indeed, certain physical aspects are inherited from the topologically quantized winding number of a so called Skyrmion, which will be defined very shorty. The methods used to test the predictions of this model exploits the Quantum Hall Effect (QHE). Basically after applying a critical depinning current, to the Skyrmions a emergent field will arise which has measurable affects on the magnitude of the Hall conductance.

## 1.1 What are Skyrmion's?

Over 50 years ago Tony Skyrme, a high-energy physicist, developed the mathematical theory which we use to describe the so called Skyrmion; in his paper, "A Unified Field Theory of Mesons and Baryons," [Nucl. Phys. 31, 556 (1962)](Fun fact: a special thanks to A.J Leggit is found at the end of this paper).

Using the concept of particles in Quantum Field Theory (wavelike excitations of the vacuum state) Skyrme attributed the stability of hadrons as being a topological defect in a Quantum Vector field[7]. As we know (or better will know), topological defects are robust; under external perturbations, such as after a continuous deformation of a vector field, the topological integer describing our defect remains unchanged. The idea that hadrons are topological defects is not commonly used by particle physicists, but condensed-matter systems have provided us with skyrmion-like topologies many years after Skyrme's paper.

Figure 1: A spintexture representing a skyrmion[7]



Let's start by describing what we mean by skyrmion. Here we avoid technical definitions like: "A skyrmion is a topologically nontrivial soliton solution of the nonlinear sigma model"[3]. We will just understand them to be point like ( 2 nm) regions of space possessing a reversal/anti-parallel magnetization relative to surrounding regions (Figure 1) [7].

The core is of finite size since the magnetization needs to unwind continuously into the background magnetization direction (see Figure 1). These patterns may be classified by a so called "winding number" which is a topological invariant.

Before going into the topology of a skyrmion, let's discuss what type of systems skyrmions are found in; we need to know about Helimagnetic structures. Helimagnetism is a peculiar ordering of magnetic structure arising from a sort of competition between different exchange interaction's. More precisely, these magnetic systems experience a dominate Heisenberg exchange interaction between neighboring spins as well as a

[1]When the magnetization of our sample does not align with and external magnetic field

Dzyaloshinkii-Moriya (DM) interaction which is attributed to electrons
feeling a spin-orbit coupling [7]; the former favors parallel alignment while the latter prefers perpendicular alignment. As a result of the competition between these two exchange interactions, there is a helical pattern exhibited by the magnetic moments. The helicity of the structure can be characterized by a relative angle $\phi \in [0, \pi]$ between consecutive spins; note ferromagnetism and anti-ferromagnetism are both special cases of Helimagnetism!

The resulting helical structure breaks spatial inversion symmetry about a plane since these twists may either be right or left handed; that is, there is a form of chiral symmetry. This motivates the concept of chiral magnets. Skyrmoins are observed in certain systems with this chiral symmetry.

The previously alluded to topological properties of skyrmions gives physicists an ambitious view for applications. Since the skyrmion is topologically protected, on the nano-scale and takes very little energy consumption (small currents) to transport, it has been hypothesized as a future means of information storage in our computers [7]. However, because we are physicists rather than engineers, we will be more interested in the apparent emergence of a kind of electrodynamics found in these systems!

### 1.2 Skyrmion Lattice Phase

Below a critical temperature, there is a phase transition analogous to that of a material transitioning from a paramagnetic phase to a ferromagnetic phase. We will take this as a given in this essay, but reference [11] gives the interested reader some systems which exhibit this phase transition. Skyrmion lattice phases (SLPs) in chiral magnets consists of topologically protected vortex lines (i.e. the Skyrmions) with a non-zero winding number (still need to define). SLPs have been found in MnSi and other B20 transition metals, and are stabilized parallel to an applied magnetic field[1].

### 1.3 Outline of what remains

A winding number has been referenced several times, but what is it? We know, as a sort of trivia fact, it has topological significance, but what does topology have to do with physics? This essay will attempt to motivate the use of topology as a powerful tool used to describe our world, but will attempt to do so with more intuitive pictures. The notion of a topological number will be illustrated through the Quantum Hall Effect (QHE) since this serves as a better starting point than the winding numbers.

After addressing the importance of the so called Chern numbers in describing our QHE, and showing the form of the winding number, we will then look at the dynamics of an electron travelling through a magnet with a spin texture. What we find is there is a link between the electron's dynamics, which we attribute to an emergent electrodynamics, and the winding number of the SLP; thus, our emergent electrodynamics inherits rules our winding number satisfies.

Finally, we look at some experiments, and show how our emergent fields can be accounted for.

## 2 Introduction to Topological Quantum Numbers

As previously stated, the QHE illustrates the use of topological quantum numbers in condensed matter systems. We choose this as our starting point for both historical as well as pedagogical reasons, but it also tells a nice story. So, the following sections may seem to be a digression from the emergent electrodynamics, but is present to motivate the use of topological quantum numbers in condensed matter systems.

## 2.1 The Quantum Hall Effect

In 1878 Edwin Hall established an experiment used to discredit one of Maxwell's statements present in his famous "Treatise on Electricity and Magnetism"; the exert states[2]: "It must be carefully remembered that the mechanical force which urges a conductor... acts, not on the electrical current, but on the conductor which carries it" Sending a current through a thin gold layer Hall was able to measure the deflection of the current carries themselves when placed in a magnetic field using a galvanometer needle. This monumental discovery established a simple yet reliable method to classify the conductor one has; you just measure the Hall voltage $V_H$. Using these concepts and the applied current, $I$, the notion of a Hall conductance was established to be:

$$\sigma = \frac{I}{V_H}$$

In 1980, nearly 100 years after Hall's discovery, Klaus von Klitzing discovered a quanta based off this almost century old concept. When we perform a Hall measurement at low temperatures and with a large magnetic field, the Hall conductance of a 2D electron gas exhibits plateaus [2]; absurdly it was found that :

$$\sigma = n\frac{e^2}{h} = \frac{n}{25812.807572\Omega}$$

Here $n \in Z$. This quantization is in terms of the ratio $G_o = \frac{e^2}{h}$ which was later deemed the conductance quantum.

## 2.2 Emergence of Topological aspects

The discovery of a new quanta is very exciting, but physicists were originally apprehensive. A moments thought reveals the peculiarity of these results: Despite microscopic imperfections, different geometries, and different concentrations of electrons dissimilar systems exhibit the same behaviour! This is, as condensed matter physicists say, a robust feature of these systems which initially troubled physicists. However, as is often the case, there's a silver-lining; these plateaus and the resulting qualms of physicists were the gateway to discovering a very general principle for Condensed Matter systems with far reaching implications.

The seemingly non-identical systems acting in a identical matter is as unintuitive to our visceral senses as, say, a Topologists idea that a coffee mug is just a doughnut. The latter two examples are linked through topological invariants, and are said to be homeomorphic to one another. As we will see, the former examples can also be classified in terms of topological invariants which take on integer values; we coin these topological invariants as as topological quantum numbers.

The notion of topological invariants; such as, Chern numbers and their non-abelian cousins, Fredholm indices, are ubiquitous. Our Hall conductance will be shown to be proportional to the first Chern number [6,5].

## 2.3 Bloch Electron's in a Uniform Magnetic Field

We are going to need the concept of a magnetic unit cell (magnetic Brillouin zone), and the generalized Bloch Bloch conditions (these are derived in detail in Appendix I). The essential information is in the presence of a Magnetic field the translation operators along the crystals lattice directions do not generally commute; indeed:

$$\hat{T}_{\vec{a}}\hat{T}_{\vec{b}} = exp(2\pi i\phi)\hat{T}_{\vec{b}}\hat{T}_{\vec{a}}$$

When our external field is rational (i.e. of the form $p/q \ni p, q \in Z$), we can enlarge our unit cell so that the the magnetic flux in the above equation, $\phi$, is an integer which defines our magnetic unit cell. Using the concept of simultaneous eigenstates, one can derive the generalized Bloch conditions;

just follow the same spirit as you would to find the Bloch conditions. For convenience we state the resulting wave function:

$$\psi^{(n)}_{k_1k_2} = e^{i(k_1x+k_2y)}u^{(n)}_{k_1k_2}(x,y)$$

and the generalized bloch conditions:

$$u^{(n)}_{k_1k_2}(x+qa,y) = e^{-i\pi py/b}u^{(n)}_{k_1k_2}(x,y)$$

$$u^{(n)}_{k_1k_2}(x,y+b) = e^{i\pi px/qa}u^{(n)}_{k_1k_2}(x,y)$$

What we can deduce from these conditions is there is an overall phase change picked up by the wave function as we travel around the magnetic unit cell boundary [Appendix I]. Indeed, if $\theta(x,y)$ is the phase of the wavefunction at each point on the boundary we find:

$$p = -\frac{1}{2\pi}\int d\vec{r}\cdot\frac{\partial\theta(x,y)}{d\vec{r}}$$

This term is a topological feature of our system, and represents the number of times an arrow parameterized by $\theta$ rotates as we go around the boundary [6]. Thus, we may think of this number as the total vorticity. Note this gives a constraint on the wave function, and it does not care what the potential is! It is, however, dependent on the external magnetic field.

## 2.4 Kubo's Linear response Formula

We will now employ a well now formula, known as the Nakano-Kubo formula, to calculate the Hall conductance. This formula is a Quantum statistical linear response calculation; that is, how a expectation value changes from its unperturbed value when a perturbation is turned on. It can be derived using the interaction picture of time dependent perturbation theory by keeping the propagator to linear order.

The details of how the Kubo formula relates to our scenario is left for Appendix II. All we stipulate here is that the Fermi energy lies within a gap which ultimately leads us to a quantized Hall conductance [5,6]. We find that the contribution to the conductance from the $n^{th}$ filled band is given by:

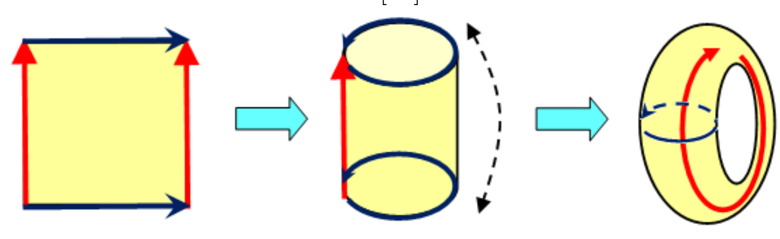
$$\sigma^{(n)}_{xy} = \frac{e^2}{2\pi i\hbar}\int d^2k[\nabla_k \times \hat{A}(k_1,k_2)]_3$$

where:

$$\hat{A}(k_1,k_2) = \int d^2r u^*_{k_1k_2}\nabla_k u_{k_1k_2} = \langle u_{k_1k_2}|\nabla_k|u_{k_1k_2}\rangle$$

An important point is that the integration is performed over the magnetic Brillouin zone, but the Magnetic Brillouin zone is topologically a Torus (see Figure 2).

Figure 2: Parallel arrows represent a periodicity; these are folded onto one another[14]



indicates this via the magnetic cells periodic boundary conditions, and folding the cell on top of itself. The identification of the magnetic Brillouin zone as a torus means the base space for $\hat{A}(k_1,k_2)$ is non-contractible allowing the possibility of a non-trivial $\hat{A}(k_1,k_2)$ [6]. This means Stoke's Theorem does not apply; thus, our conductance above may be nonzero! In a basic sense this is due to the branch cuts (which depends on the number of filled energy bands) of the wave function; the phase can not be uniquely determined throughout the entire magnetic cell [5].

### 2.5 Connections to Topological invariants

There is a lucid argument found Mahito Kohmoto's paper (reference 6) on pgs. 349-350 which considers a calculation for the simplest of non-trivial cases for $\sigma_{xy}^{(n)}$: a single zero in the unit cell for $u_{k_1k_2}$. Although easy to follow, the argument takes time, and space to set up, so we quote a few key points. We can partition the Brillouin zone into two regions: one containing the zero and one not. Using these regions and the fact the overall phase of a single component $|u_{k_1k_2}\rangle$ acquires a phase mismatch when we approach the boundary of these regions we an show our $\hat{A}(k_1, k_2)$ inherits the non-trivial topology of $|u_{k_1k_2}\rangle$. From this we can apply Stoke's Theorem to both regions individually, and this leads to a equation showing the total vorticity of the wave function in the magnetic Brillouin zone leads to $\sigma_{xy}^{(n)}$ being quantized.

The topological structure of these equations has a strong resemblance to a principle $U(1)$ bundel over a sphere $S_2$[5,8]. The $\hat{A}(k_1, k_2)$ above can be used to define a 1-form connection [6] which in turn can be used to define a curvature. In terms of our $k_1's$ and $k_2's$ the first Chern form is given by a integral of the curvature over the entire torus:

$$C_1 = \frac{i}{2\pi}\frac{\partial \hat{A}_\mu}{\partial k_\mu} dk_\mu \wedge dk_\mu$$

The entity in the integrand is, in the language of differential geometry, the curvature. a Comparison with our equation for the Hall conductance shows:

$$\sigma_{xy}^{(n)} = -\frac{e^2}{h} C_1$$

This reads the contribution to the Hall conductance from a single filled band is an integral multiple (since the Chern number is an integer) of the conductance quanta!

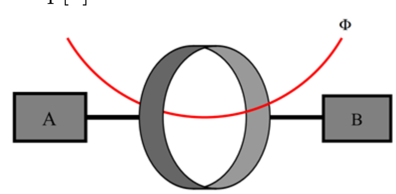
### 2.6 Laughlin's Argument

The topological aspect of the Hall conductance is important, but perhaps a bit unintuitive. Here we attempt to show how geometry and topology can be related to one another, and how geometric descriptions can arise in unexpected ways.

In 1981 Robert Laughlin introduced the seminal concepts which lead to the theory of the Integer Hall Effect[2]; basically he thought of the Hall effect as a sort of pump whose job is to transfer charge from one electron reservoir to another.

Consider a 2D electron gas confined to a cuff threaded by some magnetic flux (Figure 3). Using:

Figure 3: A Hall Pump[2]




$$\sigma_H = \frac{j}{E(r,t)} = \frac{I}{\partial_t \Phi}$$

and, if we take the time varying magnetic flux to change by a flux quanta, $\Phi_o = h/e$, in a time $\delta t$, then we get the following accumulation of charge due to this pump:

$$\Delta Q = I\Delta t = \sigma_H \Delta t \partial_t \Phi = \sigma_H \frac{h}{e}$$

Now a basic tenet of Quantum Mechanics stipulates the difference in the number of particles which accumulated on the edges must be an integer (basically particle numbers are integers); because of this, we see:

$$\sigma_H = \Delta Q \frac{e}{h} = \frac{ne^2}{h}$$

A remark made by Laughlin [2], "...by gauge invariance, adding $\Phi_o$ maps the system back to itself.... [resulting in] the transfer of n electrons." So, this was a cute, and comparatively short argument for the quantization of the Hall conductance, but clearly the rigor is missing. We take this a few steps further than Laughlin did, and recover the rigor (to some degree).

## 2.7 Modern Geometry and Curvature

It is clear that the time varying magnetic flux serves as the agent who drives charge, so this lead Laughlin to think of the QHE as a sort of pump for electric charge. The important point is pumps have cycles; a cycle of the pump corresponds to a change of the magnetic flux by a single unit of the quantum of magnetic flux[1,2]! We can use this thought process to show that the QHE can be thought of as a sort of curvature.

In 1917 Tulio Levi-Civita [2] developed a modern concept of curvature from the notion of parallel transport. The rigorous mathematical definition of parallel transport uses the notion of vector bundles and connections which is far too formal for our purposes, so we opt for intuition instead. Essentially, parallel transport involves displacing a vector about a surface (manifold) with the constraint the vector remains parallel to it's original orientation measured in respect to the surface's local tangent vector.

We are interested in abstract vectors that live in a Hilbert space, so the transportation will involve cycles in parameter space all varied adiabatically. With the notion of parallel transport we can thus view curvature as a measure of the mismatch of the phase of a system after accomplishing parallel transport about a closed circuit (This esoteric notion of curvature is illuminated when one considers the Foucault pendulum; see supplements if needed).

The important point is we have a cycle associated with our Hall pump, so Laughlin's argument motivates the Hall effect as being seen as a sort of curvature. Our mismatch in phase depends on a contemporary discovery to Laughlin: the Berry phase.

## 2.8 Berry Phases

In 1981 Michael V. Berry [9] made a discovery which was bound to happen sooner or later (and it is somewhat surprising that it wasn't sooner) known as a Berry phase. When a Quantum system is evolving adiabatically, there are two accumulated phases which arise; one due to the energy of the system, and one due to the "geometry". What do we mean by geometry though? As we have already alluded to, the Hall conductance can be thought of, in a sense, as a sort of curvature. The curvature is defined as the mismatch in phase between the initial and final system after cyclic motion in parameter space; this is what we attribute to the geometry of the system.

Instead of pursuing this example in full rigor, let's just cut to the chase. We can describe our Hall pump with the following parameters $\Phi$ and $\theta$ where $\theta$ is the phase associated with a gauge transformation of twisting the two reservoirs, the Berry phase which is accumulated from a cycle of our Hall pump can be shown to be proportional to the area of the loop. The proportionality factor is the adiabatic curvature[2]:

$$K = 2Im\langle \partial_\Phi \psi | \partial_\theta \psi \rangle$$

We will sum up our findings in the next section by simply showing how geometry ties into topology

## 2.9 Geometry and Topological Invariants

Here we borrow a theorem which links geometry with topology being the Gauss Bonnet formula[2]:

$$\int_S dAK = 2(1-g)$$

Here $K$ is the curvature of the surface $S$ at the area element $dA$ where $K^{-1}$ is defined as the product of the two local radii of curvatures (Gaussian curvature). Let's not get too hung up on the formalities here, but rather note for any surface, being geometric, there exists a topological invariant, $g$, which identifies the number of handles an object has, when we integrate over the Gaussian curvature at each point. The important point is $g$ does not change under deformations! For a doughnut and a coffee mug $g = 1$ in either case.

The integral Quantum Hall Effect uses a theorem a step above this: the Gauss-Bonnet-Chern formula. This is used for systems which need more information than what is on the surface, and the Chern number is the analog to $g$ in the above equation.

This concludes our discussion on what role topological quantum numbers play in the QHE They helped condensed matter physicists understand the conductance plateaus, and have found there way into many other descriptions of matter as we will soon see.

# 3 Back to Skyrmions

Quite a lot has been said in the last section about topological numbers, and, again, the purpose was to motivate topology as a useful concept in condensed matter physics. We are now interested in the Winding number (or the Topological Charge or the Pontryagin index), but unfortunately the mathematics is vastly greater and more complicated than that of the Hall Effect. Our discussion will thus be bare bones; we are mainly interested in the form of the winding number. In what follows we will be following Eduardo Fradkin [10].

Skyrmions are present in systems with nonuniform spin textures, and a model from Quantum Field Theory known as the nonlinear sigma model may be employed to describe spin fluctuations for an order parameter field $\vec{m}$ living in $S_2$ (surface of a sphere). The name of the game is calculate and identify the contribution to the action from various terms involved in a Hamiltonian which describes a ferromagnet. We find, after a boat load of formalism, for the Lagrangian density (after integrating out spin):

$$\mathscr{L}(\vec{m}) = \frac{1}{2g}\Big(\frac{1}{v_s}(\partial_o \vec{m})^2 - v_s(\partial_1 \vec{m})^2\Big) + \frac{\theta}{8\pi}\epsilon_{\mu\nu}\vec{m}\cdot(\partial_\mu \vec{m}\times\partial_\nu \vec{m})$$

Here $v_s$ is the spin-wave velocity and $\theta$ and $g$ are coupling constants; we are concerned with the last term.

Let's not get too buried in the technical details; instead, let's think of this as trivia (even though it's not trivial!). We define the winding number as:

$$\frac{1}{8\pi}\int d^2\vec{x}\epsilon_{ij}\vec{m}\cdot(\partial_i \vec{m}\times\partial_j \vec{m})$$

We will think of this number as the number of times the order parameter $\vec{m}$ has wrapped around the sphere $S_2$.

The important points we need for what follows is, for one, the form of this term, it is topological (invariant under continuous deformations and thus any other order parameter fields which can be continuously deformed into ours will have the same topological charge), and it is quantized. We now start what we set out to accomplish from the start!

# 4 Emergent Electrodynamics

The purpose is to show how the emergent field the electron feels inherits the quantization of the winding number given in the previous section.

## 4.1 Physical Picture

Let's consider an electron traveling through a SLP, then since the local magnetization at each point varies, the electron continuously feels a force which attempts to reorient the electrons spin in the direction $\hat{n}(\vec{r},t) = \vec{M}/|\vec{M}|$[1,7] where $\vec{M}$ is the local magnetization. If this is done adiabatically, we may describe the topological contribution of the Hall signal in terms of the Berry phases picked up by the electron[3]. The Berry phase can then conveniently be rewritten in terms of an effective Aharonov-Bohm phase if we define an emergent magnetic and electric field, $\vec{B}^e$ and $\vec{E}^e$[1,3].

Thinking of the Berry phase as the solid angle acquired by the electron as it traverses the SLP we see this depends on $\hat{n}$ since the electron has been assumed to align adiabatically with the local magnetization. As a result, the Berry phase depends on the underlying texture of the SLP which is described in terms of our topological winding numbers! Our emergent fields may be thought of as a measure of the solid angle of an infinitesimal loop in space ($\vec{B}^e$) and space-time ($\vec{E}^e$)[1]. Furthermore, recall our definition of curvature; this seems to say these fields are a measure of some sort of curvature in our SLP. The previous digression on curvature and modern geometry was present to give meaning to this statement.

## 4.2 Simple Mathematical Derivation

Here we follow an essay [4] which provided, in my opinion, the most intuitive approach. To begin we use a simple construct known as the Stoner model which pertains to spin-$\frac{1}{2}$ particles moving through a smoothly varying magnetic structure (we use the $\hat{n}$ above to describe the orientation of the local Magnetization). Our Hamiltonian takes the form:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 I - J\vec{\sigma}\cdot\hat{n}(\vec{r},t)$$

Thus, our Schrodinger equation takes the form:

$$i\hbar\partial_t\psi(\vec{r},t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 I - J\vec{\sigma}\cdot\hat{n}(\vec{r},t)\Big]\psi(\vec{r},t)$$

We can find an emergent vector and scalar potential by simply applying a local unitary transformation which rotates the spin quantization axis parallel to $\hat{n}$; this way we do not need to worry about all three components in our Pauli Matrix.

Choosing the z-axis as our (original) quantization axis we consider rotations about the following axis: $\hat{a}(\vec{r},t) = \hat{z}\times\hat{n}/|\hat{z}\times\hat{n}|$. The alignment of the quantization axis will be different for different regions of space-time, so we define the requisite angle of rotation to be $\theta(\vec{r},t)$. Thus, we introduce the following expression to our Schrodinger equation above:

$$U(\vec{r},t) = exp\big(-i\frac{\theta(\vec{r},t)}{2}\vec{\sigma}\cdot\hat{a}(\vec{r},t)\big)$$

$$\psi(\vec{r},t) = U(\vec{r},t)\phi(\vec{r},t)$$

We find:

$$i\hbar\partial_t\phi(\vec{r},t) - q^eV^e\phi(\vec{r},t) = \Big[\frac{(\vec{P}-\vec{A}^e)^2}{2m} - J\sigma_z\Big]\phi(\vec{r},t)$$

With the definitions:

$$V^e = -\frac{i\hbar}{q^e}U^\dagger\partial_t U$$

$$\vec{A}^e = -\frac{i\hbar}{q^e}U^\dagger\nabla U$$

These are our emergent potentials! The emergent charge here, $q^e$, are just taken to be plus or minus 1/2 depending on which band the electrons are in. We find from our emergent vector and scalar potentials:

$$\vec{B}_i^e = \epsilon_{ijk}\partial_j \vec{A}_k^e = \frac{\hbar}{2}\epsilon_{ijk}\hat{n}\cdot\left(\partial_j\hat{n}\times\partial_k\hat{n}\right)$$

$$\vec{E}_i^e = -\partial_i V^e - \partial_t \vec{A}_i^e = \hbar\hat{n}\cdot\left(\partial_t\hat{n}\times\partial_i\hat{n}\right)$$

Note the emergent magnetic field may be recognized as our winding number density. Thus, there are ties between this emergent field and the topology of the system, and our emergent fields inherit the quantization rules of our SLP!

### 4.3 Recap

The basic idea goes as follows: Magnetic Skyrmions are classified via a topological invariant; the "winding number". Without going into too much detail the topological winding number is responsible for inducing one unit of magnetic flux per Skyrmion [1,7], so a moving Skyrmion induces a electric field. Moreover, Faraday's law of induction inherits the topological quantization of this winding number [3] which is then used to quantitatively measure this supposed emergent electrodynamics via the Hall effect.

## 5 Experimental Details

Detecting the motions of skyrmions is a delicate subject, but since a moving skyrmion induces an emergent electric field, and these fields are proportional to the skyrmion's velocity, we can measure these induced fields to deduce the velocity of a moving skyrmion [1]. The Hall effect can be employed to make such a measurement.

Here we finish up with the apparatus, the data, and the relevant equations to the experiment (placed here to help explain some of the data).

### 5.1 Methods and Equipment

The samples used in this experiment were MnSi and they were grown using optical float-zoning under ultrahigh-vacuum compatible conditions [11]. Briefly, this technique grows bulk samples by tuning various experimental variables; such as, the feed rod, the growth rate, the atmosphere and gas pressure, the temperature gradient within the sample, the molten zone temperature and the rotation rate[12].

Each of these samples were oriented according to the diffraction patterns of a Laue X-ray experiment; the sample needs to be oriented in a known manner so that the internal magnetic field is oriented in a desired direction relative to the applied electric current. The measurements of the resistivities (Hall and longitudinal) were done via a modified six-terminal phase-sensitive detection system (basically a way of uncovering a single lost in noise like a small Hall signal).

### 5.2 Hall resistance in a S.L.P.

Not to be confused with the skyrmion Hall Effect, but the effects a SLP has on the Hall Effect.

Say our previously defined spin texture is described by $n(\vec{x}-\vec{v}_{||}t)$, with $\vec{v}_{||}$ defined as motion along the current direction, then we recognize

this as spin texture in motion. A moving spin texture actually induces an electric field [3]:

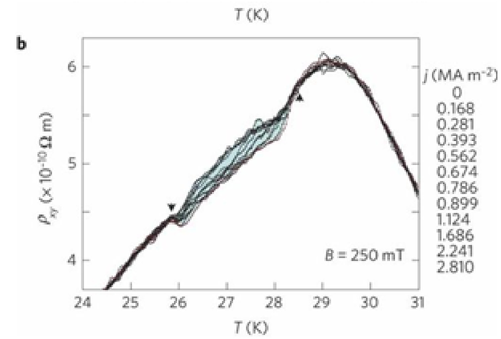
$$\vec{E} = -[\vec{v}_{||} \times \vec{B}]$$

This is analogous to the electric field induced by a moving magnetic flux! Indeed,

the magnetic structure is what is in motion, but now there are quantization rules which govern how much flux quanta can occur due to the topology of the system. Aligning the sample so that $\vec{B} = B\hat{z}$ we find an approximate form for the Hall conductivity [3]:

$$\frac{\Delta\sigma_{xy}}{\sigma_{xx}} \approx -\frac{x}{2S+x}\frac{e\langle b_z\rangle\tau}{mc}$$

Figure 4: Hall Conductance as a function of temperature for various current densities, but fixed magnetic field[1]

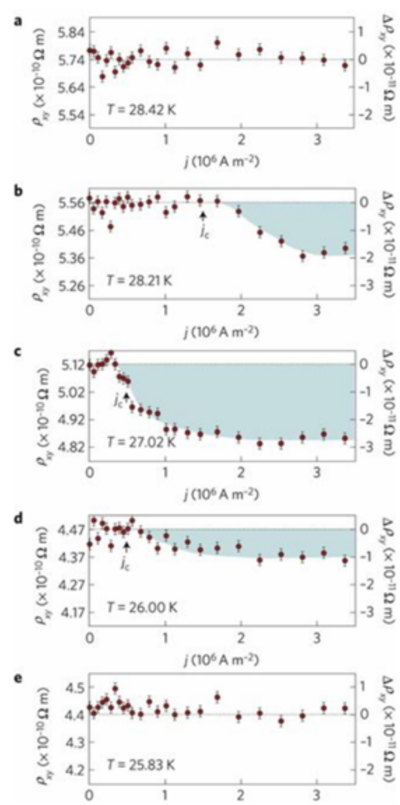



Here $m$ is the mass of the electron, $\tau$ is the relaxation time, and $\langle b_z\rangle = \frac{Q\Phi_o}{A}$ (A is the area of the SLP unit cell).

The above relation is very similar the Topological Hall effect [3]:

$$\frac{\sigma_{xy}^{TOP}}{\sigma_{xx}} \approx -\frac{e\langle b_z\rangle\tau}{mc}$$

Figure 5: Extrapolated Data [1]




The result of all this is the net Topological Hall voltage will be suppressed by a factor of $\frac{2S}{2S+x}$. Figures 4 and 5 illustrate this result. Note a critical electric current density, $\vec{j}_c$ needs to be present to overcome the pinning force [2]

of the Skyrmion lattice as shown in Figures 4 and 5 [3].

Indeed, the arrows touching the plot indicate the temperature range where the SLP is present. Figure 5 is extrapolated data from the data in Figure 4; note that the Hall conductance is suppressed when the current exceeds $\vec{j}_c$ only if the SLP is present as indicated in both figures.

A simple physical reconciliation is available; the Lorentz force on the electrons due to the internal field takes the form:

$$\vec{F} = -e[(\vec{v}_{\sigma\vec{k}n} - \vec{v}_{||}) \times \vec{B}]$$

That is, the relative velocity of the electrons in the nth band with spin orientation $\sigma$ and momentum $(\vec{v}_{\sigma\vec{k}n})$ and the skyrmions is reduced.

## 5.3 Working equations

The total force acting on an electron with momentum $\vec{k}$ and spin orientation $\sigma$ [1]:

$$\vec{F}_{\sigma\vec{k}} = e\vec{E} + \vec{F}_H + q_\sigma^e(\vec{v}_{\sigma\vec{k}n} - \vec{v}_{||}) \times \vec{B}^e$$

The additional electric current induced by the emergent field, $-\vec{v}_{||} \times \vec{B}^e$, has to be cancelled exactly by the change of the electric Hall field

[2] This pinning force is a result of varying spin direction in a SLP as well as spatial fluctuations of impurities (charge, defects, etc).

[1] $\Delta E_{\perp} = \Delta\rho_{xy} j = -\Delta\rho_{yx} j$ with $\Delta\rho_{xy} = \rho_{xy}(j) - \rho_{xy}(0)$. For a current in the x direction and a magnetic field in the z direction we find:

$$\Delta E_{\perp} \approx -\frac{\Delta\sigma_{yx} E}{\sigma_{xx}} = -\frac{\Delta j}{\sigma_{xx}} = P\left|\frac{q^e}{e}\right|(\vec{v}_{||} \times \vec{B}^e)$$

Here $P$ is a spin polarization which can be obtained from Kubos formula; basically it describes cross-correlation between the emergent current and the charge current[1].

$$P \approx -\frac{\Sigma_{n,\vec{k},\sigma=\pm} \sigma\tau_{\sigma n}(v^y_{\sigma\vec{k}n})^2_{\epsilon} f^o_{n\sigma}}{\Sigma_{n,\vec{k},\sigma=\pm} \tau_{\sigma n}(v^x_{\sigma\vec{k}n})^2_{\epsilon} f^o_{n\sigma}}$$

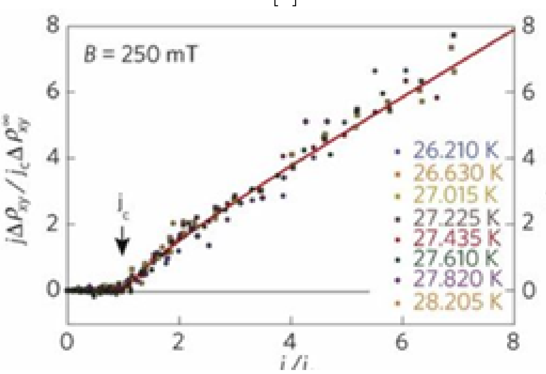
We are now in a position to approximate our emergent fields.

## 5.4 Final Results and Analysis

The groups final results consists of predictions for an emergent electric field from the measurements of the Hall conductance. To estimate the strength of the emergent magnetic field one can take into account the geometry of the skyrmion lattice since $\vec{B}^e$ is proportional to the winding number density of the skyrmion lattice.

The MnSi sample in its SLP happens to form a hexagonal lattice perpendicular to the stabilizing field[13]. knowledge of the geometry of the system allows one to calculate the windingnumber density, and hence the magnitude of $\vec{B}^e$. We will not go into detail here, and just state this can be done. Referencing our working equations we see that the measurement of the Hall field, $\Delta E_{\perp}$ gives us our emergent electric field and the drift velocity up to a factor of $P$.

Figure 6: Measurement of the transverse Electric field[1]




We need a means of estimating the drift velocity of the electrons since $v_d = E^e/B^e$. The paper uses a bunch of technical arguments found in the supplementary material, to show that the parallel component of the drift velocity is proportional to $\Delta\rho$ and hence our Hall field. Figure 6 shows, in scaled units (also explained in detail in the supplementary material), that after the application of a critical current density $\vec{j}_c$ a change of the Hall conductance is observed which we attribute to an emergent electric field.

Notice the predictions (linear plot) are better for currents just above $\vec{j}_c$, but larger values of our current density gives a larger deviation of experiment with theory. We can possibly attribute this to an eventual neglect of the pinning force for sufficiently large current densities which results in a greater conductance, but that is more speculation than anything. The important points is the predictions are good and knowledge of the drift velocity can be gleaned from the Hall measurement, and thus the emergent electric field. Indeed, we have $E^e_{\perp} = v_{d||} B^e$[1].

# 6 Conclusions

The papers "finale" came across as somewhat anticlimactic. The authors seemed to justify the importance of all the figures explicitly till the end. Yes, there are relationships to find our emergent electric field given Hall data in a SLP, but I would of hoped for more of a discussion. For instance, what makes thinking of this Hall data as an emergent field advantageous to another thought process? The speculated usefulness of an emergent electrodynamics is in predicting the motions of skyrmions, but I do not see any explicit reasons why this is easier. Just seems like the build up to the measurement could have used more discussion. The discussion on some useful aspects were presented, but in spirit of theory rather than experiment (e.g. the accumulation of the Berry phase can be thought of as an Aharonov Bohm effect if we introduce our emergent fields).

Even though I wish there was more of a discussion on the experimental verification of the predictions that a fictitious emergent field provides, it is stated these fields are useful in calculating the dynamics of skyrmions; thus, these concepts may be useful in the future as a means to manipulate and/or detect magnetic whirls. Skyrmion's have been hypothesized as potentially being used as nano-scale memory units, microwave oscillators and logic elements [1]. The appeal lies in the size of the defect (2 nm), but also the exceedingly low current densities needed to displace the magnetic skyrmions.

In my eyes this seems a bit ambitious. The selective temperature ranges SLP exist in are far too restrictive for the small depinning current density to compensate for, and I believe it will take years to find a practical use. This does not mean we should not pursue research though; it's just time is needed.

I want to briefly comment on some assumptions of this paper. First, a non-uniform spin texture can be thought of as arising from a different number of electrons filling a spin band (majority and minority spin bands). We assumed the electrons also move through the structure adiabatically which is apparently not that bad of an assumption[4], but there was no account taken for scattering processes which may result in a spin flip of some kind. This would seem to have some sort of effect on the magnetic structure of the SLP, so should maybe be cared for.

A quick interesting point; after 150 years of its discovery, the Hall effect still finds novel applications in classifying properties of matter. For instance the Hall effect is sensitive to motions in the SLP, as we have seen; hence, we can use it to classify the critical depinning velocity and which SLP may be most useful in future applications. It is neat how a simple idea can illuminate such a diverse range of physics.

The paper overall was very interesting, and I learned a lot digging around in research papers. The use of topological quantum numbers is new to me, and I will be learning more over the Summer in a reading course. I am not too sure where I'd be able to take this next, but it has been speculated swapping magnetic vortices satisfies a non-abelian algebra. Braids from algebraic topology are used to describe this. Possibly these emergent fields can be used to clearly describe how to swap our vortices.

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# 8 Appendix I

***We follow reference [6] throughout. Here we look at the topological aspects of an electron's wavefunction subject to a periodic potential, and an external magnetic field. Most of the concepts will be taken from [6]. The main point of this subsection is to establish the connection between the linear response formula (i.e. Kubo) and the Hall conductance which can then be related to topological invariants via the Chern numbers.

Before the realization of topological defects, it was assumed the phase of the wave function could be determined all around the magnetic Brillouin zone, but this is only true if and only if there is no Hall current [6]. On the surface this may seem like a problem, but as is often he case in physics, there is a silver lining. We can use this concept to describe the discretization of the Quantum Hall effect.

It was realized that the integral values of the Hall conductance were achieved, being multiples of $e^2/h$, if the Fermi energy lies in a gap between Landau levels [6]. Here we show that this quantization has a geometric origin, and establishes a very general principle. Let's consider a system of non-interacting electrons in 2D immersed in a uniform magnetic field, then Schrodinger's equation takes the form:

$$H\Psi = \big[\frac{1}{2m}(\hat{p} + e\hat{A})^2 + U(x,y)\big]\Psi = E\Psi$$

With: $U(x+a,y) = U(x,y+b) = U(x,y)$. We note the system is invariant about certain translations, but the Hamiltonian may not respect these symmetries because of the magnetic vector potential (even though the field is uniform!). To circumvent this slight issue we can start by introducing the Bravais lattice vectors:

$$\vec{R} = n\vec{a} + m\vec{b} \quad \ni n, m \in Z$$

and the translation operator for a given Bravais lattice vector:

$$T_{\vec{R}} = exp\{(i/\hbar)\vec{R}\cdot\vec{p}\}$$

Then we note in the presence of a uniform magnetic field $\vec{B}$ using the symmetric gauge we have $\vec{A} = \frac{\vec{r}\times\vec{B}}{2}$. We then define the magnetic translation operator [13]:

$$\hat{T}_{\vec{R}} = T_{\vec{R}} exp\{(i/\hbar)\vec{R}\cdot(\vec{p} + e\frac{\vec{r}\times\vec{B}}{2})\} = T_{\vec{R}} exp\{(ie/2\hbar)\vec{r}\cdot\vec{B}\times\vec{R}\}$$

The beauty of this operator is it leaves the Hamiltonian invariant! That is:

$$[\hat{T}_{\vec{R}}, H] = 0$$

We may now proceed forward as if we are proving Bloch's Theorem, but with a few important caveats. For one the magnetic Translation operators along distinct lattice vectors do not, in general, commute with one another, but instead:

$$\hat{T}_{\vec{a}}\hat{T}_{\vec{b}} = exp(2\pi i\phi)\hat{T}_{\vec{b}}\hat{T}_{\vec{a}}$$

Here $\phi = (eB/h)ab$. Thus, if $\phi = p/q$, where $p$ and $q$ are relatively prime, then a certain subset of translations commute with one another. We use this property to define a magnetic unit cell, and ultimately the magnetic Brillouin zone by enlarging the unit cell so that an integral multiple of magnetic flux penetrates this new cell. For example, if we take our rational number above, then:

$$\vec{R_p} = n(q\vec{a}) + m\vec{b}$$

Then we have $p$ units of magnetic flux penetrating this magnetic unit cell, and more importantly these translation operators commute with one another! Thus, we have a set of three mutually compatible observables, so we can apply the same prescription that Bloch himself used. For eigenfunctions $\psi$ of $H$ we see:

$$\hat{T}_{q\vec{a}}\psi = e^{ik_1 qa}\psi$$
$$\hat{T}_{\vec{b}}\psi = e^{ik_2 b}\psi$$

where: $k_1 \in [0, 2\pi/qa]$ and $k_2 \in [0, 2\pi/b]$. We should immediately realize at this point the wavefunction takes the form (using $n$ as the band index):

$$\psi^{(n)}_{k_1k_2} = e^{i(k_1x+k_2y)}u^{(n)}_{k_1k_2}(x,y)$$

We can show (using equations.....) that $u^{(n)}_{k_1k_2}(x,y)$ satisfies the following properties:

$$u^{(n)}_{k_1k_2}(x+qa,y) = e^{-i\pi py/b}u^{(n)}_{k_1k_2}(x,y)$$
$$u^{(n)}_{k_1k_2}(x,y+b) = e^{i\pi px/qa}u^{(n)}_{k_1k_2}(x,y)$$

These are known as the generalized Bloch conditions, and they have VERY IMPORTANT consequences. Note a translation of the wave function about the boundary of the magnetic Brillouin zone results in an accumulation of phase. Before moving on, let's highlight this in following way. Let $\hat{L}$ be a translation operator about some path, thought of as infinitesimal translation), then we note:

$$\hat{L}u^{(n)}_{k_1k_2} = |u^{(n)}_{k_1k_2}|exp\big[i\theta_{k_1k_2}(x,y)\big]$$

Furthermore, we note (as shown in H.W. 6):

$$\vec{A}' = \vec{A} + \nabla\Lambda \Rightarrow \psi' = e^{-\frac{ie}{\hbar}\Lambda}\psi$$

Because a gauge transform changes the phase of the wave function, we are interested in the accumulated phase a wavefunction acquires after traversing the boundary of the magnetic Brillouin zone. We know as a result of our magnetic field and boundary of our magnetic unit cell:

$$p = \frac{-1}{2\pi}\int d\vec{l}\cdot\frac{\partial\theta_{k_1k_2}(x,y)}{\partial\vec{l}}$$

A moments thought about the above result should be, initially, puzzling; the accumulated phase does not depend on the external potential! This means the total vorticity in the magnetic unit cell of the wave function is a topological invariant depending on the magnetic field.

# 9 Appendix II

Here we introduce our linear response equation. To begin let's case the Schrodinger equation into a more suitable form:

$$\hat{H}(k_1,k_2)u^n_{k_1k_2} = E^n u^n_{k_1k_2}$$

Here:

$$\hat{H}(k_1,k_2) = \frac{1}{2m}\big(-i\hbar\nabla + \hbar\vec{k} + e\vec{A}\big)$$

If we are considering a physical situation with a sufficiently small electric field, then we may use the Nakano-Kubo linear response formula [6]. Thus, the Hall conductance is given by:

$$\sigma_{xy} = \frac{e^2\hbar}{i}\Sigma_{E^n<E_f<E^m}\frac{[v_y]_{nm}[v_x]_{mn} - [v_x]_{nm}[v_y]_{mn}}{(E^n - E^m)^2}$$

Here we stick to our earlier stipulation that the Fermi energy lies in between two bands. Sparing the knitty-gritty details we find for our matrix elements:

$$[v_i]_{nm} = \frac{1}{\hbar}\langle n|\frac{\partial \hat{H}}{\partial k_i}|m\rangle$$

$$= \frac{1}{\hbar}(E^m - E^n)\langle n|\frac{\partial u^m_{k_1k_2}}{\partial k_i}\rangle = \frac{1}{\hbar}(E^n - E^m)\langle\frac{\partial u^n_{k_1k_2}}{\partial k_i}|m\rangle$$

Substituting this back into our conductance formula, using the identity $\Sigma_{E^n<E_f<E^m}\Big[||n\rangle\langle n| + |m\rangle\langle m|\Big]$, and defining:

$$\hat{A}(k_1,k_2) = \int d^2r u^*_{k_1k_2}\nabla_k u_{k_1k_2} = \langle u_{k_1k_2}|\nabla_k|u_{k_1k_2}\rangle$$

We can write the contribution of the nth band to the Hall conductance as:

$$\sigma^{(n)}_{xy} = \frac{e^2}{2\pi i\hbar}\int d^2k\big[\nabla_k \times \hat{A}(k_1,k_2)\big]_3$$

# 10 Supplementary

## 10.1 Foucalt

Borrowing the adiabatic theorem from classical mechanics we know that the plane of the Foucault pendulum is fixed in space, relative to the local tangent plane of Earth, which means a vector contained in this plane is being parallel transported on a closed loop. This transportation is due to the rotational velocity of Earth about its own axis. The well known geometric phase this vector acquires is just the solid angle subtended by the path taken at the specific latitude the pendulum is located; that is, $\Omega = 2\pi cos(\theta_o)$ where $\theta_o$ is the angle from the polar axis.

This simple example illustrates the curvature of the Earth since the solid angle does depend on the radius.