Half-Quantum vortices in a two-dimensional chiral p-wave superconductor

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Abstract.

In this term paper we give a brief introduction to the half-quantum vortices (HQVs) in a two-dimensional chiral p-wave superconductor such as Sr_2RuO_4. HQV was not observed in Sr_2RuO_4 in experiments for years, which results from the unstability of HQV in bulk material. Until most recently, Budakian's group observed half-height magnetization steps in micron-size samples of Sr_2RuO_4, which indicates the existence of HQVs. In this term paper we discuss the stability of HQV and give an introduction to the experimental results. In addition, we introduce zero-energy Majorana fermions staying in the cores of HQVs, which lead to the non-Abelian statistics of HQVs. Based on this characteristic, we give a short discussion on other proposals to detect HQVs.

My interest in choosing this topic originates from: (i) Prof. Goldenfeld's discussion on topological defects in his lecture. (ii) The seminar in which Prof. Budakian reported the observation of half-fluxoid state in mesoscopic samples of Sr_2RuO_4 in Nov. 2010.

1. Introduction

Most known superconductors are characterized by spin-singlet paring, while Sr_2RuO_4 is characterized by spin-triplet paring [1]. This extra spin freedom allows for the existence of half quantum vortex (HQV). Actually, any multi-component order parameter allows for fractional quantum vortex. Now let's take an example, for an order parameter with only one freedom, e.g., $\Delta = \Delta_0 e^{i\phi}$, the order parameter can come back to itself after $\phi \to \phi + 2\pi$. However, for an order parameter with two freedoms, e.g., $\Delta = \vec{d}e^{i\phi}$, the order parameter comes back to itself after $\vec{d} \to -\vec{d}$, $\phi \to \phi + \pi$. This the basic physics for HQV and we will illustrate this with a concrete example of HQV in Sr_2RuO_4 below.

Another thing is why do we investigate HQV in a two-dimensional (2D) superconductor? Because it is predicted that HQVs in (quasi-) two-dimensional superfluids will have Majorana fermion zero modes bounded at vortex cores, which satisfy non-Abelian statistics [2] and thus can be utilized to make topological quantum computation [3]. Actually, HQVs were first proposed to exist in A-phase of superfluid ^{3}He [4]. However, it is very difficult to make a sufficiently thin 2D film of ^{3}He, while a 2D solid film is much easier to fulfill in experiments. As for the underlying physics of HQV in ^{3}He and Sr_2RuO_4, they are the same. If any difference, ^{3}He is neutral while Sr_2RuO_4 is charged. Here we mention that whether the system is charged or not affects the stability of vortex directly. For example, a full quantum vortex is stable in bulk superconductor because its dynamic energy is finite due to the screened effect (Messner effect), while in superfluid there is no such screen effect, and as a result the vortex's energy is divergent with the size of sample.

The third thing is can we observe HQVs in experiments? For years HQVs were not observed in Sr_2RuO_4, which reminds us to investigate on the stability of HQVs. As we mentioned, a full quantum vortex is stable in bulk superconductor because of screen effect. However, the physics underlying HQVs and full quantum vortices are not the same, because HQV is caused not by charge current but spin current, which is not screened. This means the dynamical energy of a single HQV is divergent with the sample's size [5]. Therefore, to stabilize a HQV, the best method is to limit the sample's size. It is under this direction that Budakian's group found evidence for the existence of HQV in micron-size Sr_2RuO_4 rings most recently [6]. In their work, HQV was indirectly validated by observing 'half-integer transition' between fluxoid states. It is noticed that the non-Abelian statistics of HQVs is not utilized in this experiment, and we may ask: can we find other schemes to observe HQVs by utilizing non-Abelian statistics of HQVs? We will give an introduction to some proposals based on this idea and discuss how to make topological quantum computation with HQVs.

The structure of this term paper is arranged as follows: First we introduce HQV starting from the order parameter of p-wave superconductor, and discuss its stability in both bulk and finite-size samples. Next we introduce the experimental results which validate the existence of HQVs. Then we discuss the physics of Majorana fermions which exist in the vortex core, based on which we introduce other proposals to observe

HQVs.

2. Basics physics for HQV and its stability in $\mathbf{p}_x$+$\mathbf{ip}_y$ wave superconductors

2.1. Introduction to HQV

For spin-triplet superconductors, the order parameter has the form:[1]

$$\Psi = e^{i\theta}[\ (d_x + id_y)|\uparrow\uparrow\rangle + (d_x - id_y)|\downarrow\downarrow\rangle + d_z(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\ ](k_x + ik_y). \quad (1)$$

Notice that the order parameter has two freedoms: one is the overall phase θ, and the other is the direction $\vec{d}$ of the triplet paring. HQV exists because $\vec{d}$ can change sign when circulating around the vortex core, while θ changes by π simultaneously. As a result, the order parameter in Eq.(1) can keep unchange.

Now we suppose the direction of $\vec{d}$ can only rotate in $x-y$ plane, then the order parameter can be further simplified as:

$$\Psi = \Delta(r)(e^{i\phi}|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)(k_x + ik_y), \quad (2)$$

from which we can find the up-spins and down-spins are decoupled. From this point of view, when circulating around the vortex core, ϕ is changed by 2π. That's to say, up-spins see a full single-quantum vortex, while down-spins see no vortex. (We can also say down-spins see a full single-quantum vortex while up-spins see no vortex, by writing the oder parameter in the form $\Psi = \Delta(r)(|\uparrow\uparrow\rangle + e^{i\phi}|\downarrow\downarrow\rangle)(k_x + ik_y)$.) Therefore, *HQV is essentially caused by spin current.*

2.2. Stability of HQV in p_x+ip_y wave superconductors

We start from discussing the stability of conventional full quantum vortices. For simplicity, we first discuss the neutral case, e.g., vortices in neutral superfluid. For superfluid with order parameter $\Delta(r) = \Delta_0 e^{i\phi(r)}$, the velocity of supercurrent is of the form $v(r) \sim \nabla\phi(r)$, where r is the distance from vortex core. From the defination of vortex $\oint \vec{\nabla}\phi(r)d\vec{l} \sim n$, where interger n is called winding number, we can get $v(r) \sim 1/r$. Supposing the length of the vortex core (in which $\Delta(r) = 0$) is $\sim \xi$, and that of the system is $\sim L$, we can get the dynamical energy of the system is $\sim \ln(L/\xi)$, which means the energy is divergent with the system's size L. Therefore a single full vortex is not stable in this case (Fig.1(a)).

What's the reason for this divergent energy? It comes from the requisite $\oint \vec{\nabla}\phi(r)d\vec{l} \sim n$ even if the radius of the closed path is infinity ($r \to \infty$). How to resolve this divergence? Apparently there are two methods: one is to limit the system's size, i.e., $r < L$; and the other is to make $n = 0$. For the former method we just need to make the sample's size as small as possible. For the later method, we know it is of no sense for a single quantum vortex with $n = 0$. However we can make $\sum n_i = 0$, with each $n_i \neq 0$. The simplest case are a pair of vortices with $n_1 = 1, n_2 = -1$ (Fig.1(b)). For a closed bath involving these two vortex cores we have $\oint \vec{\nabla}\phi(r)d\vec{l} \sim (n_1 + n_2) = 0$, which resolve the divergence problem. (Here it is beneficial to make an analogy with a pair of charges with $\pm q$, and most of the electric field lines start from $+q$ and end at $-q$, but not end at infinity.) For a pair of vortices with opposite winding number, we can obtain the dynamical energy in the form $\sim \ln(R/\xi)$, where R the distance between two cores. Therefore, from the point of energy favored, we get the conclusion that (i) in neutral superfluid a single vortex can be stable only when the system's size is finite; (ii) a pair of vortices with opposite winding numbers can be stable. (For quantitative calculation of the vortices' energy, please refer to Ref [5].)

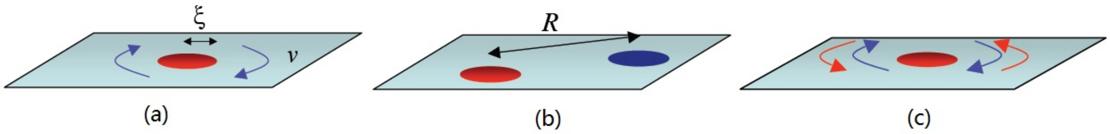

Figure 1. (a) A single quantum vortex with core length ξ in neutral superfluid. The dynamic energy is divergent with the system's size L in the form $E \sim \ln(L/\xi)$. (b) A pair of vortices with opposite winding number. The energy divergence is canceled in the form $E \sim \ln(R/\xi)$, where R is the distance between two cores. (c) A single full quantum vortex in charged superfluid. The energy divergence is canceled by screen effect (Messner effect).

Next we consider a single full quantum vortex in charged superfluid. In this case the charged superfluid is coupled with gauge field. From the minimal coupling principle the supercurrent velocity is of the form: $v(r) = (\hbar/m)\nabla\phi(r) - (2e/c)\mathbf{A}$, where the extra term $\sim \mathbf{A}$ (vector potential) leads to screen effect. Now we go along a closed path around the vortex core: $\oint \vec{\nabla}\phi(r)d\vec{l} \sim \oint \mathbf{A}d\vec{l} + \oint \vec{v}(r)d\vec{l} = 2\pi n$. (Here I may miss some constants such as c, $\hbar$, $2e$ and so on.) In a concise form, $\Phi' = \Phi + \oint \vec{v}(r)d\vec{l} = 2\pi n$. In this case it can be easily shown that for $r \to \infty$, $\oint \vec{v}(r)d\vec{l} = \Phi' - \Phi \to 0$. That's to say, the energy divergence is canceled because of screen effect. This screen effect comes from the coupling between charge current and gauge field (Fig.1(c)). Therefore, a single full quantum vortex is stable in charged superfluid.

Finally, we discuss the stability of HQV in a p-wave superconductor. As we mentioned in 2.1, HQV is caused by spin current but not charge current, which means there is no screen effect for HQV, i.e., a single HQV is not stable in bulk material. Therefore, the stability of HQV in charged superconductor is similar to that of a single full quantum vortex in neutral superfluid. Then we can conclude that there are two ways to get stable HQVs, one is to limit the size of materials, and the other is to get a pair of HQVs. These two methods are exactly those Chuang et al. proposed in their article [5], which also suggested various mesoscopic geometries to stabilize and observe HQVs. Here we emphasize that if one pair of HQVs with opposite winding numbers are stable, it is totally probable that a lattice of HQVs can also be stable. Various structures of stable HQV-lattice were calculated by Chuang et al [7].

3. Evidence of HQV in experiments

As we discussed above, one can obtain stable HQVs by limit the size of smaples. It was based on this scheme that Budakian's group made micron-size Sr_2RuO_4 rings and detected the supercurrent therein (The photo of such micron-size Sr_2RuO_4 ring is shown in Fig.2).

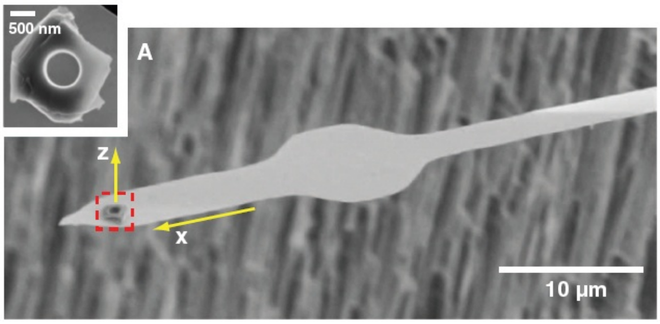

Figure 2. Image of a Sr_2RuO_4 ring on a cantilever. From J. Jang *et al.*, 2011.

First we give an introduction to the principle in this experiment. For simplicity, we start from the detection of a single full quantum vortex. As we discussed in Sec.2, the supercurrent velocity is $\vec{v}(r) = (\hbar/m)\vec{\nabla}\phi(r) - (2e/c)\vec{\mathbf{A}}$. By integrating around the ring and considering the density of superfluid, one can get $\oint \vec{\nabla}\phi(r)d\vec{l} \times \Phi_0/2\pi = \oint \vec{\mathbf{A}}d\vec{l} + (4\pi/c)\oint \lambda^2 \vec{j}_s d\vec{l} = n\Phi_0$, where Φ_0 is the flux quantum and λ is the penetration depth. In a concise form, $\Phi' = \Phi + (4\pi/c)\oint \lambda^2 \vec{j}_s d\vec{l} = n\Phi_0$, i.e., the fluxoid Φ' is quantized. Notice that Φ is the flux through Sr_2RuO_4 ring and can be controlled by external magnetic field $\vec{H}$. From the equations above, on can easily find that with $\vec{H}$ increasing, the supercurrent $\vec{j}_s$ will decrease linearly, and then the magnetic moment induced by $\vec{j}_s$ will decrease linearly too. When external magnetic field increase to certain value, the number of fluxoid will jump from n to $n+1$. Accordingly, the supercurrent $\vec{j}_s$ will make a jump, leading to a jump magnetic momentum, as shown in Fig.3.

Then we discuss what would happen if a HQV appears in this experiment. As for the full quantum vortex, we know that the vortex is caused by a charged current, i.e., both spins-up charges ($\uparrow\uparrow$) and spins-down charges($\downarrow\downarrow$) contribute to the vortex, which can be simply expressed as $n = (n_{\uparrow\uparrow} + n_{\downarrow\downarrow})/2$. (Notice that in this model there is no $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$ paring.) For HQV vortex, as seen from Eq.(2), either (i) $|\uparrow\uparrow\rangle$ sees a full quantum vortex and $|\downarrow\downarrow\rangle$ sees no quantum vortex, or (ii) $|\uparrow\uparrow\rangle$ sees no quantum vortex and $|\downarrow\downarrow\rangle$ sees a full quantum vortex. That's to say, when a jump happens, either $(n_{\uparrow\uparrow}, n_{\downarrow\downarrow}) \to (n_{\uparrow\uparrow}+1, n_{\downarrow\downarrow})$ or $(n_{\uparrow\uparrow}, n_{\downarrow\downarrow}) \to (n_{\uparrow\uparrow}, n_{\downarrow\downarrow}+1)$. In either case we have $n \to n+1/2$. Therefore, the jump height of magnetic momentum corresponding to a HQV should be one half of the jump height corresponding to a full quantum vortex. This conclusion was verified in Budakian group's experiments, as shown in Fig.3. They observed transitions between integer fluxoid states as well as half-integer transitions, the height of which is half of the ones they observed between integer fluxoid states.

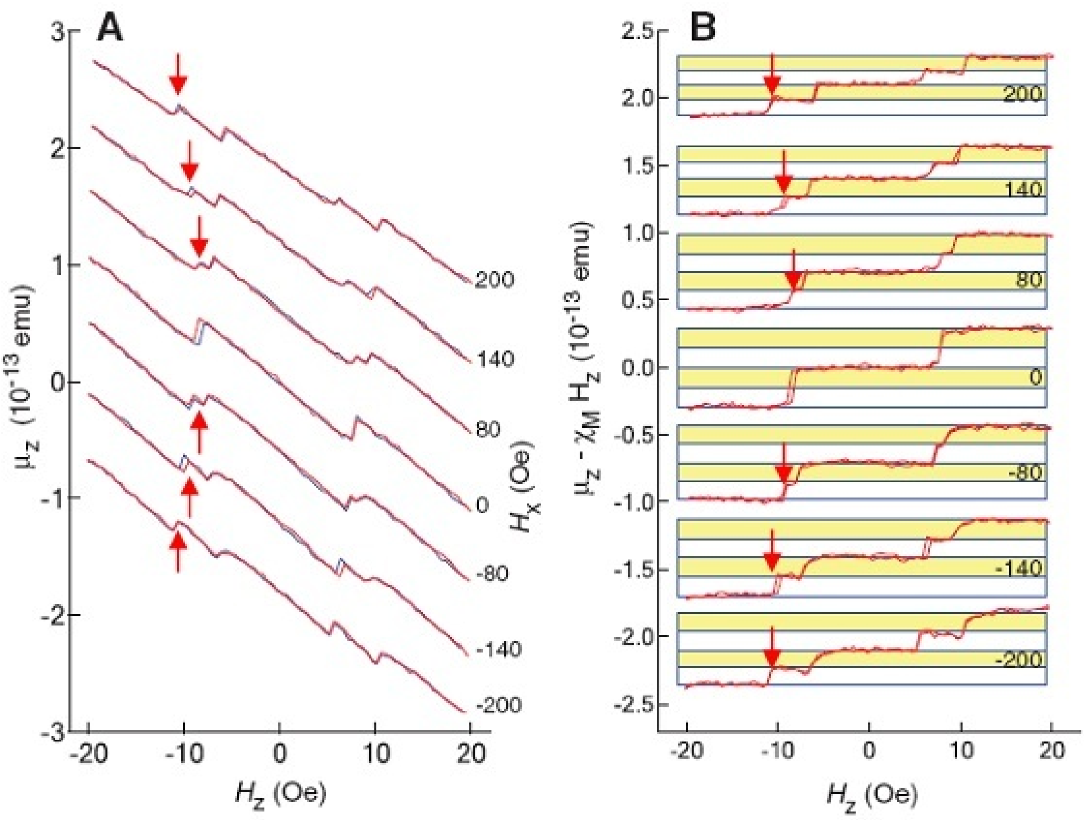

Figure 3. (a) Evolution of magnetic momentum induced by supercurrent with increasing external magnetic field. Both integer transitions and half-integer transitions were observed. The positions where half-integer transitions happen are marked by red arrows. (b) Meissner response was subtracted. From J. Jang *et al.*, 2011.

4. Basic physics for Marjorana Fermion

Now we have introduced the basic physics for HQV and its stability, as well as the demonstration of HQV in experiments . What's more amazing is the existence of Majorana fermions in HQV cores. Next we derive the existence of Majorana fermions from the Hamiltonian of p-wave superconductors. The mean field Hamiltonian for such superconductors is

$$H = \int d^2r[\Psi^\dagger(-\frac{\vec{\nabla}^2}{2m} - E_f)\Psi + \Psi^\dagger[\Delta(\vec{r}) * (\vec{\nabla}_x + i\vec{\nabla}_y)]\Psi^\dagger + h.c.], \quad (3)$$

where E_f is the fermion energy and $*$ is the symmetrized product ($a * b = (ab + ba)/2$). This Hamiltonian may be diagonalized by solving the corresponding BdG equations,

$$\begin{pmatrix} -\vec{\nabla}^2/2m - \mu & i\{\Delta(\vec{r}), \vec{\nabla}_x + i\vec{\nabla}_y\}/2 \\ i\{\Delta^*(\vec{r}), \vec{\nabla}_x - i\vec{\nabla}_y\}/2 & \vec{\nabla}^2/2m + \mu \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = E \begin{pmatrix} \mu \\ \nu \end{pmatrix}. \quad (4)$$

The Hamiltonian then takes the form:

$$H = E_0 + \sum_n E_n \gamma^\dagger \gamma \tag{5}$$

with $\gamma^\dagger = \mu\Psi^\dagger + \nu\Psi$ being the generating operator of Bogoliubov quasiparticles. From the analysis in section 2.1 we know that up-spins and down-spins are decoupled. Therefore, when we solve BdG equation, the equations are divided into two pieces †. One for up-spins and the other for down-spins. The piece for down-spins has no relation with vortice. However, the piece for up-spins is equivalent to that of a full single-quantum vortice in spinless p-wave superconductor. Then we come to the key point: *a HQV for spinful fermions is equivalent to a single-quantum vortex in a p-wave superconductor of spinless fermions.*

As discussed in Ref [8], in spinless p-wave superconductors, Bogoliubov quasiparticles corresponding to low-energy spectrum are a linear superpositions of generating and annihilating operators of the *same* species. Now we come back to HQV for spinful fermions, considering the equivalence between two cases, we can obtain:

$$\gamma_i^\dagger = \mu\Psi_{i\uparrow}^\dagger + \nu\Psi_{i\uparrow}, \tag{6}$$

where $\Psi_{i\uparrow}^\dagger$ is the electron generating field for up-spins at the core of the ith vortice. This relation is very interesting, considering in conventional superconductors quasiparticles correspond to a linear superpositions of generating and annihilating operators of fermions with different spins. This difference leads to an additional relation between positive and negative energy eigen-states:

$$\gamma_i^\dagger(E) = \gamma_i(-E), \tag{7}$$

from which we can obtain the interesting relation $\gamma_i^\dagger(0) = \gamma_i(0)$, which is the defination of Marjorana fermion operator. In this case, the particle is its own antiparticle and Eq.(6) becomes $\gamma_i^\dagger = \mu\Psi_{i\uparrow}^\dagger + \mu^\star\Psi_{i\uparrow}$. In addition, it is easy to find that we need two sets of Marjorana fermion operators to construct one set of fermion operators in the way $c^\dagger = \gamma_1 + i\gamma_2$ and $c^\dagger = \gamma_1 - i\gamma_2$ to satisfy $\{c, c^\dagger\} = 1$. In other words, the degrees of freedom in a HQV is half of that of a single quantum vortex in a conventional superconductors.

To sum up, the underlying physics of why Marjorana fermion can exist in HQV is: In HQV, the freedom of spin is excluded (considering that only up-spins can see a whole single-vortice while the down-spins see no vortices.) and so we can fulfill a spinless case. Therefore, the mix of generating($\Psi^\dagger$) and annihilating(Ψ) operators can correspond to the same spin. In other words, $\Psi^\dagger$ and Ψ have no relationship with spins, which make the self-conjugate situation $\gamma^\dagger = \gamma$ be possible.

† Here I will not repeat the detailed calculating process. The calculation process to divide BdG equation into two pieces is trivial. However, to solve the piece with a single-quantum vortex is very interesting. For details you can refer to Ref. [8]. This calculation process will help to understand the underlying physics.

5. Non-Abelian Statistics of Half-Quantum Vortices

Here we quote A. J. Legget's \$64K question in his lecture:†
What Berry phase does the Majorana fermion acquire when the gap rotates through 2π*?*

We can get the answer by resolving BdG equation with $\Delta(\vec{r}) \to \Delta(\vec{r})e^{i\Omega}$, and it is easy to get

$$\gamma_i^\dagger \to \mu e^{-i\Omega/2}\Psi_{i\uparrow}^\dagger + \mu^* e^{i\Omega/2}\Psi_{i\uparrow}. \tag{8}$$

As a result, when Ω is changed by 2π, $\gamma \to -\gamma$. The underlying physics lies in the fact that $\Delta(\vec{r})$ is the order parameter of particles with charge 2e, while the quasi-particle is a linear superposition of fermion creation and annihilation operators carrying charge 1e. Then we raise another interesting question: *What happens if the* i*th HQV circulates around the* j*th HQV?*

When there are many well separated HQVs(labeled with i) at positions $\vec{R}_i$, the gap function near the ith HQV has the form $\Delta(\vec{r}) \propto |\Delta(\vec{r})|e^{i\Omega_i}$, with $\Omega_i = \sum_{j\neq i} \arg(\vec{R}_j - \vec{R}_i)$. Then when the ith HQV circulates around the jth HQV, $\arg(\vec{R}_j - \vec{R}_i)$ changes 2π, which is independent on the geometry trajectory that the ith HQV takes. Thus both Ω_i and Ω_j change by 2π. According to Eq.(8), we have $\gamma_i \to -\gamma_i$ and $\gamma_j \to -\gamma_j$. That's to say, when the ith HQV circulates around the jth HQV, both γ_i and γ_j are multiplied by -1.

Notice that here the changes happen on operators. Supposing U_{ij} is the unitary transformation operating on the ground state when the ith HQV circulates around the jth HQV, it can be viewed as transforming γ_k in the form: $\gamma_k \to U_{ij}^\dagger \gamma_k U_{ij}$. Notice for $k \neq i, j$, Ω_k keeps unchange under the transformation, that's to say, γ_k keeps unchange. This requirement can fix U_{ij} up to an Abelian phase: $U_{ij} = e^{\pi\gamma_i\gamma_j/2}$.

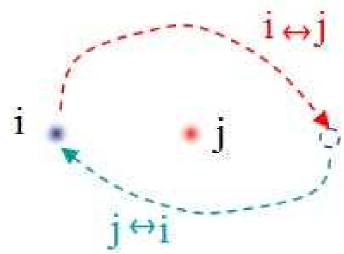

Figure 4. Scheme of circulating HQV i around HQV j is equivalent to two exchanges of HQV i and HQV j.

Then we come to the case in which we exchange positions of HQV i and HQV j. It is noticed that the effect of circulating HQV i around HQV j is equivalent to two exchanges of HQV i and HQV j, as shown in Fig.4. Then unitary transformation that corresponds to an exchange is the square root of U_{ij}, i.e.,

$$U_{i\leftrightarrow j} = \exp(\pm\pi\gamma_i\gamma_j/4). \tag{9}$$

† Please refer to A. J. Legget's Lecture note 27 in the course 'Physics in two dimensions' on website: http://online.physics.uiuc.edu/courses/phys598PTD/fall09/

One can find two such transformations do not commute if they share one HQV, i.e., the HQVs satisfy non-Abelian statistics.

6. Observe half-quantum vortices with interference effect

Considering the non-Abelian statistics of HQVs, we may ask if we can utilize this characteristic to observe HQVs. Actually, the topological properties of HQVs are identical to those of the the Pfaffian quantum Hall state, which obey non-abilian brading statistics[9]. An elegant method to observe such effect in quantum Hall effect was proposed by Fradkin *et al.* by using interference principle [10]. Considering the two systems are equivalent to each other from the point of statistics. we can directly borrow the method from quantum hall system.

This method was first borrowed here by Das Sarma *et al.* [9], and most recently Vishveshwara's group revisited this effect and combined it with Aharonov-Casher interference [11]. Considering the former method is a special case of the later one, we introduced Vishveshwara group's method here.

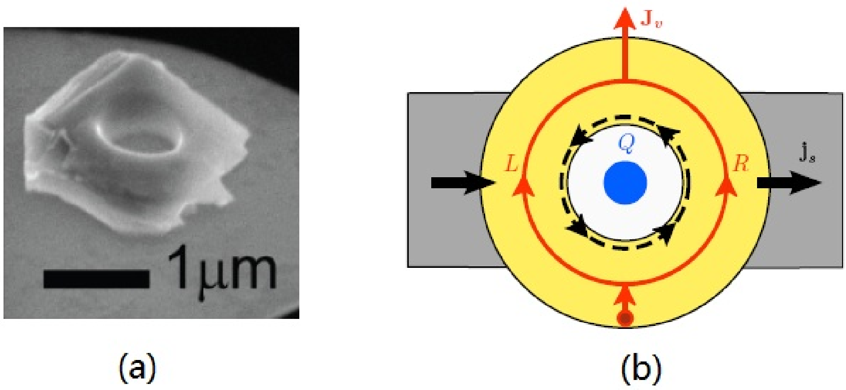

Figure 5. (a) A Sr_2RuO_4 sample used Budakian group's experiments. (b) Scheme of setup for observing HQVs in Budakian group's sample. A supercurrent j_s is maintained across the ring. Because of Magnus force, there is also a current J_v flowing from bottom to top. J_v can choose either left side of right side, which leads to interference effect. From Grosfeld *et al.*, 2010.

Shown in Fig.5 (a) is the sample used in Budakian group's experiments. To observe interference effect, we need to connect the micron-size ring to two leads, and maintain a supercurrent j_s across the ring (see Fig.5 (b)). If there are vortices in the ring, then these vortices will suffer to Magnus force and a current J_v flowing from bottom to top will appear. (To understand the Magnus force, we just need to think about the arcing-ball when we play soccer or ping-pang.) The current in J_v direction will dead to a voltage drop V which can be measured by experiments. (To understand the voltage drop, we just need to recall the Josephson relation between voltage drop and phase difference, considering current J_v is directly related with phase difference.) If a HQV enters from the bottom, then it will move upward under Magnus force, by choosing either the left path or the right path. If the motion of HQV is coherent, then interference will happen, which leads to an observable effect by measuring voltage drop. For HQVs which obey non-Abelian statistics, the vortex current J_v was first calculated by Fradkin *et al.* in the form:

$$J_v \propto |t_L|^2 + |t_R|^2 + 2|t_L||t_R| Re(e^{i\phi}\langle\psi_L|\psi_R\rangle), \tag{10}$$

where t_L (t_R) is the amplitude of wavefunction $|\psi_L\rangle$ ($|\psi_R\rangle$), and $\phi = \arg(t_L^{\star} t_R)$. Till here, it is the method introduced by Fradkin and borrowed by Das Sarma. It is found when the number N of HQVs in the hole is even, $\langle\psi_L|\psi_R\rangle$ is finite, and when N is odd, $\langle\psi_L|\psi_R\rangle$ is zero. Most recently, Vishveshwara's group put some charges Q at the hole, which can lead to Aharonov-Casher interference. A-C interference says when a fluxline circles a total charge Q, it can acquire a geometric phase. That's to say, by changing charge Q, we can change ϕ in Eq.(10) and thus lead to a periodic oscillations in the observed voltage drop, which makes the interference much clear. However, as we mentioned, if HQVs' number N is odd, $\langle\psi_L|\psi_R\rangle$ is zero, and the voltage drop will be independent on Q.

7. Conclusion

In this term paper, we made a brief introduction to HQV in a 2-D chiral superconductor such as Sr_2RuO_4. We discussed the stability of HQVs as well as the experimental results. In addition, we introduced Majorana fermions which are bounded in the cores of HQVs. The existence of Majorana fermions leads to non-Abelian statistics for HQVs, based on which we introduced other proposals to observe HQVs.

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