Quantum Phase Transition

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A quantum system can undergo a continuous phase transition (QPT) at the absolute zero of temperature as some parameters entering its Hamiltonian are varied. These transitions are particularly interesting for, in contrast to their classical finite-temperature counterparts, their dynamic and static critical behaviors are intimately intertwined. Considerable insight is gained by considering the path-integral description of the quantum statistical mechanics of a system in which time appears as an extra dimension. In particular, this allows the deduction of scaling forms for the finite-temperature behavior, which turns out to be described by the theory of finite-size scaling. In this essay, 1 dimensional Ising chain is studied as an example for QPTs. Then I briefly described QPTs in quantum hall effects and high T_c superconductors.

I. INTRODUCTION

Nature abounds with phase transitions. Usually they are classified as first order or second order. The usual argument for phase transition is based on free energy.[1] Phase transitions occur when the ground states of free energy are different for different temperatures.

Here I will describe a new kind of phase transitions– *Quantum Phase Transition*, which occurs at the absolute zero of temperature. Here the ground states vary as varying some coupling constants. Let us suppose that, for a given coupling constants g, which can be pressure, composition or magnetic field strength, etc[4], we have obtained the energy levels of the system: that is, we know the configuration that minimizes the energy, the configuration that corresponds to the first excited state of the system *etc.* We can plot the energy levels respecting to the coupling constant g (FIG 1).

If the first excited level crosses the ground level at a coupling constant g_c (FIG. 1), we

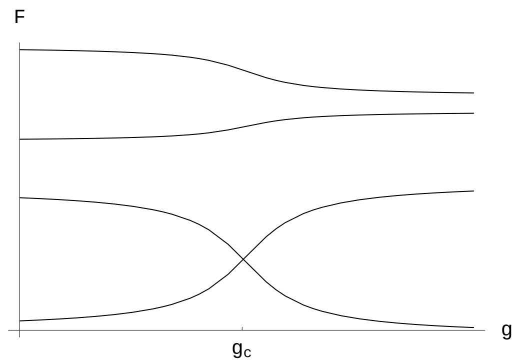

FIG. 1: Zero temperature phase transition by level crossing.

can expect a phase transition occurs when we tune the coupling constant g through this point. This kind of phase transitions is termed as Quantum Phase Transition. At the exact point of g_c, the state is decided by the initial conditions. Then why do we interested in this phenomena in absolute zero, which is impossible to happen in nature? Because associated with this critical point of phase transition, a particular region, known as quantum critical region, demonstrates some novel and deep properties, which hold the key to understanding a wide range of behaviors in many condensed-matter systems.

II. QUANTUM STATISTICAL MECHANICS

In quantum statistics, the most interesting thing is the partition function of the system, which is governed by a Hamiltonian H [5, 6],

$$Z(\beta) = \mathrm{Tr} e^{-\beta H} \tag{1}$$

The expectation of an arbitrary operator O is

$$< O >= \frac{1}{Z(\beta)} \mathrm{Tr}(O e^{-\beta H}) \tag{2}$$

where $k_B T = 1/\beta$. Notice that the operator density operator $e^{\beta H}$ is the same as the time-evolution operator $e^{iH\mathcal{T}/\hbar}$, provided we assign the imaginary value $\mathcal{T} = -i\hbar\beta$ to the time interval over which the system evolves. Thus we see that calculating the thermodynamics of a quantum system is the same as calculating transition amplitudes for its evolution in imaginary time, with the total time interval fixed by the temperature of interest. Then the Feynman's path-integral formulation of quantum mechanics can be used. Formally,

$$e^{-\beta H} = [e^{-(1/\hbar)\delta\tau H}]^N \tag{3}$$

where $\delta\tau$ is a time interval that is small on the time scales of interest ($\delta\tau = \hbar/\Gamma$, where Γ is some ultraviolet cutoff) and N is a large integer chosen so that $N\delta\tau = \hbar\beta$. Then

$$\begin{aligned} Z(\beta) = & \sum_n \sum_{m_1,m_2,\cdots,m_N} < n|e^{-(1/\hbar)\delta\tau H}|m_1 > \\ & \times < m_1|e^{-(1/\hbar)\delta\tau H}|m_2 > \times < m_2|\cdots|m_N > \\ & \times < m_N|e^{-(1/\hbar)\delta\tau H}|n > \end{aligned} \tag{4}$$

Then we can treat temperature as another dimension. This treatment is exact when $T \to 0$ and this extra "time" direction diverges. In this way, we can mapping a d dimensional quantum system into a $d+1$ dimensional classical system. It is clearer in Table I.

Quantum	Classical
d space, 1 time dimensions	$d+1$ space dimensions
Coupling Constant g	Temperature T
Inverse temperature β	Finite size L_τ in "time" direction
Correlation length ξ	Correlation length ξ
Inverse characteristic energy $\hbar/\Delta$,$\hbar/k_B T_c$	Correlation length in the "time" direction ξ_τ

TABLE I: Mapping in the statistics[6]

III. QUANTUM ISING CHAIN IN A TRANSVERSE FIELD

Most of the important concepts in QPT arise from the 1 dimensional Ising model. [2, 3] And results of this model are believed to be exact. So it is worthwhile to review it more carefully.

The Hamiltonian of 1 dimensional Ising model is

$$H_I = -J \sum_i (g\hat{\sigma}_i^x + \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z) \tag{5}$$

The quantum-classical mapping ensures that this transition will be in the universality class of the $D = 2$ classical Ising model, which is extensively studied.

Firstly, let us analyze the limit situation in $T = 0$.

Strong coupling: $g \gg 1$ the ground state is simply ferromagnetic state.

$$|0\rangle = \prod_i |\rightarrow\rangle_i \tag{6}$$

where $|{\rightarrow}\rangle_i$ and $|{\leftarrow}\rangle_i$ are eigenstates of $\hat{\sigma}_i^x$ with eigenvalues ± 1.

$$\begin{aligned} |{\rightarrow}\rangle_i &= \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle_i + |{\downarrow}\rangle_i\right) \\ |{\leftarrow}\rangle_i &= \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle_i - |{\downarrow}\rangle_i\right) \end{aligned} \tag{7}$$

where $|{\uparrow}\rangle_i$ and $|{\downarrow}\rangle_i$ are eigenstates of $\hat{\sigma}_i^z$ with eigenvalues ± 1.

The first excited states are

$$|i\rangle = |{\leftarrow}\rangle_i \prod_{j \neq i} |{\rightarrow}\rangle_j \tag{8}$$

Weak coupling: $g \ll 1$ At $g = 0$, there are two degenerate ground states.

$$|{\Uparrow}\rangle = \prod_i |{\uparrow}\rangle_i \tag{9}$$

$$|{\Downarrow}\rangle = \prod_i |{\downarrow}\rangle_i \tag{10}$$

The real state will be the linear superposition of these two states. It is the so-called quantum paramagnetic state.

The excited states can be described in terms of an elementary domain wall (or kink) excitation. For instance the state

$$\cdots |{\uparrow}\rangle_i |{\uparrow}\rangle_{i+1} |{\downarrow}\rangle_{i+2} |{\downarrow}\rangle_{i+3} |{\downarrow}\rangle_{i+4} |{\uparrow}\rangle_{i+5} |{\uparrow}\rangle_{i+6} \cdots$$

Intermediate coupling The first excited states has the energy

$$\epsilon_k = 2J(1 + g^2 - 2g\cos k)^{1/2} \tag{11}$$

The energy gap is $|\Delta|$

$$\Delta = 2J(1 - g) \tag{12}$$

In two limits, the ground states are different. Obviously there is a quantum phase transition at some crossover point g_c. From Eq. 12 we can find $g_c = 1$.

we can plot the phase diagram as Fig2. In this diagram, we have two energy scale in this diagram, Δ and T(set $k_B = 1$). We can regard them as quantum scale and thermal dynamical scale, respectively. In Regions A,B, $T \ll \Delta$, both systems can be treated as the classical systems although the Δ have different structure for each region, i.e., they have different quasi-particles. In TableII, we can find they have the same dynamics characters,

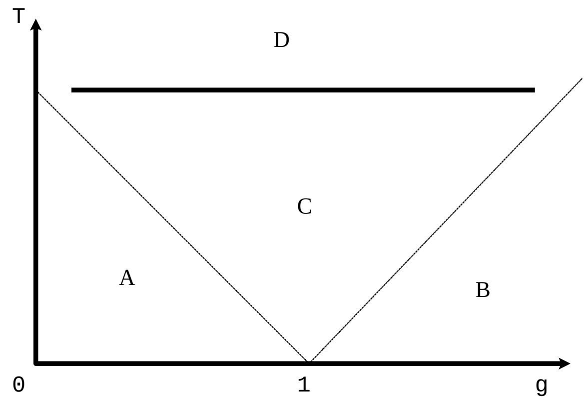

FIG. 2: Phase diagram of Ising Chain

A: Low T on the Magnetically Ordered Side, $\Delta > 0, T \ll \Delta$

B: Low T on the Quantum Paramagnetic Side, $\Delta < 0, T \ll \Delta$

C: Continuum High T, $T \gg \Delta$

D: Lattice High T, $T \gg J$ The properties in this region are nonuniversal and determined by the lattice scale Hamiltonian.

their phase coherence times share the same structure. However, Region C, Continuum high-T, has completely different properties about correlation length and correlation time. The relaxation procession here are completely quantum. We cannot account it by the classical explanation. (see Fig3) In order to get the properties of this region, we need so-called "continuum quantum field theory".

IV. QPTS IN OTHER SYSTEMS

Besides 1 dimensional Ising chain, there are a lot of systems have QPTs.

Quantum Hall Effect The two-dimensional electron gas in semiconductor heterostructures has a very rich phase diagram with a large number of quantum phase transitions. [2] Consider a particular class of transitions that are relevant to QPTs. The electronic spins in one layer are fully polarized in the direction of external field in ground state. If we bring two layers close to each other. For large layer spacing, the two layers

	Low-T (magnetically ordered). Quasi-classical particles-domain walls	Continuum high-T (quantum critical)	Low-T (quantum paramagnet). Quasi-Classical particles-flipped spins
ξ	$\left(\frac{\pi c^2}{2\Delta T}\right)^{1/2} e^{\Delta/T}$	$\frac{4c}{\pi T}$	$\frac{c}{\lvert\Delta\rvert}$
τ_ϕ	$\frac{\pi}{2T} e^{\Delta/T}$	$\frac{\cot(\pi/16)}{2T}$	$\frac{\pi}{2T} e^{\Delta/T}$

TABLE II: Correlation length, ξ(defined from the exponential decay of the equal-time correlations of the order parameter), and the phase coherence time, τ_ϕ, in the different regimes of Fig2. The two low-T regimes have an interpretation in terms of quasi-classical particles, but the physical interpretation of the two particles are very different, as indicated.[2]

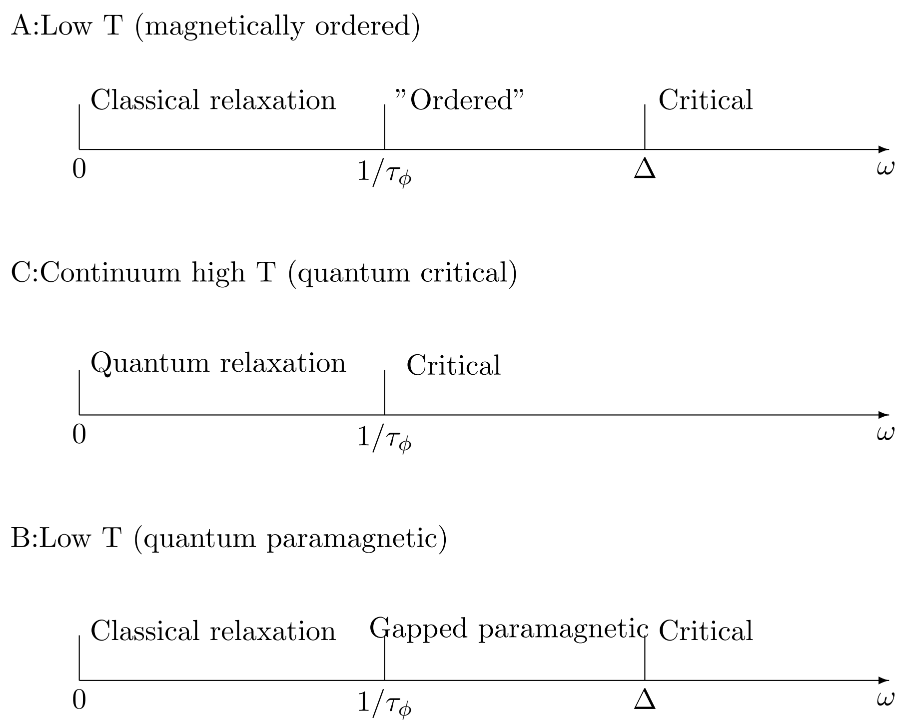

FIG. 3: Crossover as a function of frequency for the Ising model in the different regime of Fig2

will have their ferromagnetic moments both aligned in the direction of the applied field. While for smaller spacings, there turns out to be a substantial antiferromagnetic exchange between the two layers, so ground state eventually becomes a spin singlet, created by a bonding of electrons in opposite layers into spin singlet pairs. There is a quantum phase transition between these two states.

High T_c Superconductor Considering the high T_c superconductor materials $La_{1.85}Sr_{0.15}CuO_4$. [2, 4] The parent compounds of it is an insulator La_2CuO_4. Each copper ion has a single unpaired electron, and has total spin $S = 1/2$. The interactions between the electrons are antiferromagnetic and the ground state is a "Néel" state, in which the spins are polarized in opposite orientations on the two checherboard sublattices of the square lattice. The strontium ion has one less electron to donate to the copper oxide layers than lanthanum, so the effect of doping is to introduce "holes" into the square-lattice antiferromagnet. Experiments demonstrated that holes tend to concentrate in "stripes". Imagine that the spins in the hold-poor regions interact with their nearest neighbours by an antiferromagnetic exchange of energy $\mathcal{J}$, and that the spins at the boundaries of two neighbouring regions are coupled by a weaker exchange of magnitude $\lambda\mathcal{J}$, where $0 \leq \lambda \leq 1$. At $\lambda = 1$, this model reverts to the ordinary square lattice $S = 1/2$ antiferromagnet and its ground state exhibits long-range magnetic order. While ground state for $\lambda \ll 1$ is a quantum paramagnet state. Then a quantum phase transition must occur for an intermediate λ.

One thing I should point out is that, in most cases, the boundary between A and C is the real phase boundary, while there is no real phase transition between B and C.

V. SUMMARY

We have a generic picture for quantum phase transition like Fig4. The C region exhibits novel properties that we are far from fully understanding. Using the continuum quantum field theory (CQFT), we can get some results, such as correlation length and time, the scale relation about it, *which comes after renormalization and I do not understand.* Surely the study on these properties will help us to understanding behaviors of condensed matter more deeply.

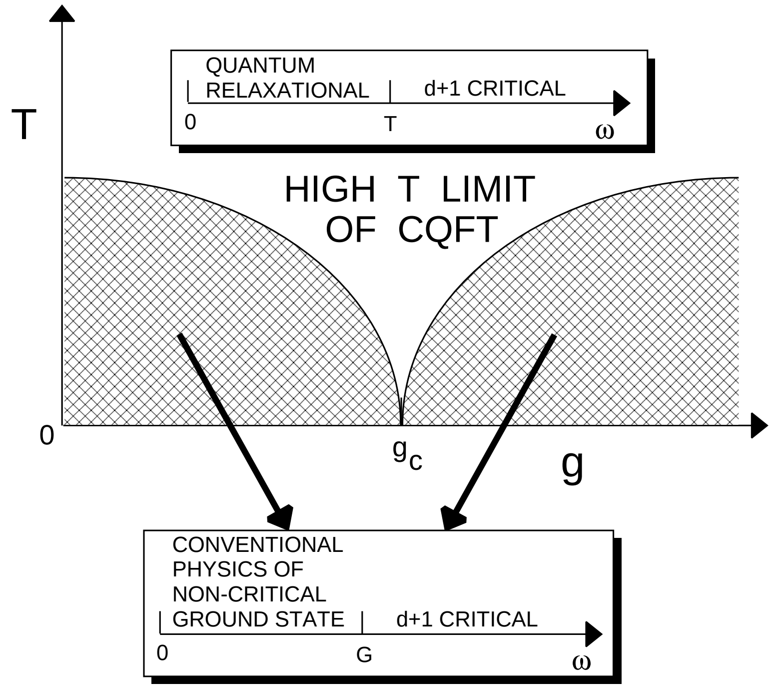

FIG. 4: Generic phase diagram

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