Chern-Simons Theory of Fractional Quantum Hall Effect

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Abstract

The Fractional Quantum Hall effect is reviewed from the perspective of Chern-Simons field theory. The interacting 2D electron gas in magnetic field problem is exactly mapped to a bosonic problem in which the bosons couple to a new gauge field in addition to the electromagnetic field. It is shown that mean field analysis in the new formulation is sufficient to explain all basic features of the fractional quantum Hall effect.

1 Introduction

The theory of Fractional Quantum Hall Effect (FQHE) provides us with a striking example of a strongly interacting system which can be well understood in terms of the weak interactions of nontrivial effective objects . What is most striking about FQHE is that due to their fractional charge the effective objects, unlike atoms and molecules, cannot be thought of as a simple collection of more elementary particles.

The FQHE is basically the problem of interacting electrons in 2D in the presence of a strong perpendicular magnetic field. The ground state wave function is completely different from the non-interacting one and the problem cannot be understood by means of perturbation theory starting from the non-interacting case. The many body ground state wave function was guessed by Laughlin with the help of the variational principle. Once this highly non-trivial step was made the fractional Hall effect theory developed quickly and many different ways of looking at the problem were invented. In this paper we will review a work done by Zhang, Hansson and Kivelson [1][2] who mapped the interacting electron problem to one of interacting bosons coupled to an additional gauge field. The advantage of the new formulation is that near filling factors of the form 1/(2k+1) the essential features of FQHE can be derived by a straightforward mean field analysis. This formulation enables us to compute quantities in a systematically improvable way using the machinery of perturbation theory.

Chern-Simons Landau-Ginzburg (CSLG) approach to the FQHE makes use of the fact that in 2 dimensions an electron can be treated as a boson to which an odd number of magnetic flux quanta are attached. For filling factors $\nu = 1/(2k+$

1) the ground state is just a homogeneous bosonic field. The topologically trivial fluctuations around the uniform state are gaped as can be easily seen from the classical equations of motions for a charged liquid. The lowest lying excitations are topological vortices in the uniform state. These vortices carry fractional charge $e/(2k+1)$. The resistive dissipation if present comes from the motion of the vortices. Under conditions in which FQHE effect is observed the vortices are pinned down by the effects of disorder just like in type II superconductors. When the vortices are free to move they behave as quasi particles with fractional charge and fractional statistics. These quasi particles can in turn condensate creating a hierarchy of FQHE states. This hierarchy can explain FQHE at filling fractions different from $\nu = 1/(2k+1)$. If the average electron density does not correspond exactly to a filling factor of $\nu = 1/(2k+1)$ the extra charge is accommodated by creating localized vortices in the otherwise uniform Bose field. This is analogous to the way in which type II superconductors accommodate extra magnetic field.

2 Experimental Observations

An effectively 2D electron system is created at the interface of a semiconductor and an insulator or two semiconductors (one of them acting as an insulator). The electrons are trapped in a quantum well in direction perpendicular to the surface formed by the insulator(acting as a high barrier) and an applied electric field perpendicular to the interface. The quantum well is narrow enough so that the z dependence of the wave function is quantized and at low temperatures the z-dependence of the wave function is fixed to the lowest level.

A current is applied to the 2D system, and the resulting Hall voltage in the perpendicular direction is measured. It is observed that for certain samples and certain applied perpendicular magnetic fields the transverse conductivity is $\sigma_{xy} = fe^2/h$ with f a rational fraction and at the same time the longitudinal conductivity $\sigma_{xx} = 0$ - both with a very high accuracy. This is a defining characteristic of the Hall effect.

Let ρ denote the electron density and B the applied magnetic field in z-direction. The plateaus are formed around filling factors $\nu \equiv \rho hc/(eB)$ which are rational fractions. The plateaus are most prominent at fractions of the type $1/(2k+1), k = 0, 1, 2, ...$

The other fundamental aspect of the fractional quantum Hall effect is the existence of fractionally charged quasi particles. The charge of the quasi particles is just the electron charge times the fraction f for the corresponding plateau. The charge of the quasi particles can be measure directly using a device called quantum antidot electrometer[3]. The most notable result is that the fractional charge is the same within a given plateau, ex. for filling factors close but not equal to 1/3 the quasi particles still have charge e/3.

3 Chern-Simons theory of FQHE

3.1 Mapping to Bosonic Problem

The microscopic Hamiltonian for a collection of electrons in external electromagnetic field $(A_0, \mathbf{A})$ is

$$H = \sum_i \frac{1}{2m}\Big[\mathbf{p}_i - \frac{e}{c}A(\mathbf{r}_i)\Big]^2 + \sum_i eA_0(\mathbf{r_i}) + \sum_{i<j} V(|\mathbf{r}_i - \mathbf{r}_j|) + g\mu\mathbf{B}\cdot\mathbf{S}, \quad (1)$$

where m is the band mass of the electrons in the crystal. V(r) is the Coloumb potential or some more general two-body interaction. The FQHE is observed for strong magnetic fields so that the electrons are almost fully polarized (in GaAs the Zeeman splitting is about 1/70 of the cyclotron energy) and we will ignore spin from now on.

The above Hamiltonian acts on the space of antisymmetric functions and defines an eigenvalue problem

$$H\psi(r_1, ..., r_N) = E\psi(r_1, ..., r_N) \quad (2)$$

By performing a unitary transformation we will map it to an equivalent eigenvalue problem for symmetric wave functions:

$$H'\phi(r_1, ..., r_N) = E\phi(r_1, ..., r_N) \quad (3)$$

The unitary transformation we will use is

$$U = \exp\big(-\imath\sum_{i<j}\frac{\theta}{\pi}\alpha_{ij}\big), \quad (4)$$

where α_{ij} is the angle between $\mathbf{r}_i - \mathbf{r}_j$ and, say, the x-axis. Since $\alpha_{ij} = \alpha_{ji} + \pi$ swapping r_i and r_j leads to a phase change $\exp(-\imath\theta/\pi)$ in U. Therefore for the special choice $\theta = (2k+1)\pi$ if ψ is antisymmetric then $\phi \equiv U^{-1}\psi$ is symmetric. In order to obtain the same eigenvalue problem but for ϕ we define $H' = U^{-1}HU$. Noticing that

$$U^{-1}\big(\mathbf{p}_i - \frac{e}{c}\mathbf{A}(r_i)\big)U = \mathbf{p}_i - \frac{e}{c}\mathbf{A}(r_i) - \hbar\frac{\theta}{\pi}\sum_{j\neq i}\nabla\alpha_{ij} \quad (5)$$

we introduce the statistical gauge operator

$$\mathbf{a}(\mathbf{r}_i) = \frac{\phi_0\theta}{2\pi^2}\sum_{j\neq i}\nabla\alpha_{ij} \quad (6)$$

and write down the new bosonic Hamiltonian as

$$H' = \frac{1}{2m}\sum_i\big[p_i - \frac{e}{c}\mathbf{A}(\mathbf{r}_i) - \frac{e}{c}\mathbf{a}(\mathbf{r}_i)\big]^2 + \sum_{i<j}V(\mathbf{r}_i - \mathbf{r}_j). \quad (7)$$

The second quantized version of H' is

$$H' = \int d^2 r \phi^\dagger \Big[\frac{1}{2m}\big(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(\mathbf{r_i}) - \frac{e}{c}\mathbf{a}(\mathbf{r_i})\big)^2 + eA_0(x)\Big] \\ +\frac{1}{2}\int\int d^2 r d^2 s(\rho(r)-\bar{\rho})V(r-s)(\rho(s)-\bar{\rho}), \tag{8}$$

where $\phi(r)^\dagger$ and $\phi(r)$ are the standard bosonic field operators, and

$$a^\alpha(\mathbf{r}) = -\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\epsilon^{\alpha\beta}\int d^2\mathbf{s}\frac{r^\beta - s^\beta}{|\mathbf{r}-\mathbf{s}|}\rho(\mathbf{s}) \tag{9}$$

is the second quantized expression for a in terms of the bosonic field operators. $\epsilon^{\alpha\beta} = \epsilon^{0\alpha\beta}$ where $\epsilon^{\mu\nu\rho}$ is the standard Levy-Civita tensor. Throughout this article α and β will run over the two space components, and the indices μ, ν, ρ will run over space-time with 0 being the time component. We included in the Hamiltonian a uniform positive background with the same mean density to avoid divergence of energy density in the thermodynamic limit.

Since U is a unitary transformation the charge number density is simply

$$\rho(x) = \phi^\dagger(x)\phi(x). \tag{10}$$

Correspondingly the number current following from the charge conservation equation

$$\partial_t\rho + \nabla\cdot\mathbf{j} = 0 \tag{11}$$

is

$$\mathbf{j} = \frac{\hbar}{2m\imath}(\phi^\dagger\nabla\phi - \nabla\phi^\dagger\ \phi - 2\imath\frac{e}{c\hbar}(\mathbf{A}+\mathbf{a})\rho). \tag{12}$$

We will later determine the expectation value of this quantity in the presence of electric field and from there the conductivity.

Although we have demonstrated the equivalence of the two problems the usefulness of the transformation is not apparent at this level. If anything, the new Hamiltonian looks more complicated. However, as we will show below, for an appropriate average electron density $\mathbf{a}$ can cancel exactly the $\mathbf{A}$ field and as a result a uniform ground state for the bosons becomes possible. By considering the Gaussian fluctuations around it the Laughlin wave function can be recovered after we transform back the wave function.

3.2 Coherent state path integral formulation

The goal of this paper is to show that in the new formulation of the problem mean field theory is a good approximation and derive some of the FQHE phenomenology from it. In order to perform the mean field analysis we will first obtain the path integral representation of the above Hamiltonian. This formulation is not required for the mean field theory but it is the best staring point if we want to study corrections to it. The recipe for constructing a coherent state path integral is to write down the second quantized Lagrangian density $\mathcal{L}(\phi, \phi^\dagger) = \imath\hbar\phi^*\partial_t\phi - H(\phi, \phi^*)$ and then formally set

$$Z[A] = \int [d\phi] \exp \frac{\imath}{\hbar} \int dt d^2 x \mathcal{L}(\phi(x), \phi^*(x)), \tag{13}$$

where the path integral is over all c-functions $\phi(x)$.

As we saw above the operator $\mathbf{a}$ is expressed in terms of the bosonic field operators. Therefore we can immediately write down the path integral over the bosonic field. However it is much more elegant to write the path integral in a way which threats a as an independent field. To accomplish this we need to add an extra term $\mathcal{L}_a$ to the action which recovers the equations of motion for $\mathbf{a}$ in terms of the bosonic field. Notice that the complicated operator identity 9 can be replaced by the following two operator equations:

$$\epsilon^{\alpha\beta}\partial_\alpha a_\beta(r) = \phi_0 \frac{\theta}{\pi}\rho(r) \tag{14}$$

$$\partial^\alpha a_\alpha(r) = 0 \tag{15}$$

These two equations can be viewed as Maxwell equations for the field $\mathbf{a}$ and (9) as the corresponding 2D Biot-Savart solution.

We will formally introduce a new time component a_0 of the field $\mathbf{a}$ and make the Lagrangian linear in it. The idea is that the integration over a_0 will produce a δ function of its coefficient. For each $\mathbf{r}$ we have

$$\int da_0(r) \exp\Big(\imath\big(\epsilon^{\alpha\beta}\partial_\alpha a_\beta(r) - \phi_0 \frac{\theta}{\pi}\rho(r)\big)a_0(r)\Big) = \delta(\epsilon^{\alpha\beta}\partial_\alpha a_\beta(r) - \phi_0 \frac{\theta}{\pi}\rho(r)) \tag{16}$$

This δ-function will enforce (14) when we perform the integration over the space components $\mathbf{a}$. Equation (15) will be enforced by restricting the path integral only to paths $\mathbf{a}^T$ which satisfy $\partial^\alpha a^T_\alpha = 0$. Putting together everything we said above we get

$$Z[A] = \int [d\phi][da^T][da_0] \exp(\frac{\imath}{\hbar} \int dt d^2 r \mathcal{L}) \text{ with} \tag{17}$$

$$\mathcal{L}(\phi, a) = \phi^*\imath\hbar\partial_t\phi - \imath\big(\epsilon^{\alpha\beta}\partial_\alpha a_\beta(r) - \phi_0 \frac{\theta}{\pi}\rho(r)\big)a_0(r) - H'(\phi, \mathbf{a}, \mathbf{A}) \tag{18}$$

It is also possible to remove the transverse gauge restriction in the path integral by the Fadeev-Popov procedure. A term $-\frac{e\pi}{2\theta\phi_0}\epsilon^{\alpha\beta}a_\alpha\partial_t a_\beta$ needs to be added to the Lagrangian. Then after regrouping we obtain the standard Chern-Simon Lagrangian $\mathcal{L} = \mathcal{L}_a + \mathcal{L}_\phi$ with

$$\mathcal{L}_a = \frac{e\pi}{2\theta\phi_0}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \tag{19}$$

and

$$\mathcal{L}_\phi = \phi^*\big(\imath\hbar\partial_t - e(A_0 + a_0)\big)\phi - H'(\phi, \mathbf{a}, \mathbf{A}). \tag{20}$$

This path integral is exact reformulation of the original problem.

3.3 Mean Field Theory

The mean field theory consists in replacing the path integral with the contribution from its most important path - the classical path which makes the action stationary. This path satisfies the classical Euler-Lagrange equations of motion. The equation of motion obtained by variation with respect to a_0 is simply (14) with the operators replaced by ordinary fields.

Let us consider the case $A_0 = 0$ (no external electric field) and $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = -B$ (uniform magnetic field). From the classical field equations for ψ and a it is easy to see that a uniform solution for ϕ is possible provided that $B = \phi_0 \frac{\theta}{\pi}\bar{\rho}$. The solution is simply

$$\phi(r) = \sqrt{\bar{\rho}},\ \mathbf{a}(r) = -\mathbf{A}(r),\ a_0(r) = 0\ . \tag{21}$$

In this state we can think of ϕ as a charged Bose condensed which is not coupled to an external potential.

If we apply electric field $E_\mu = -\partial_\mu A_0$ we can deduce the conductivity. Looking at (12), (19) and (20) we see that

$$< j_\alpha(x) >= Z^{-1}\frac{\delta Z}{\delta A_\alpha(x)} = \frac{\delta S_A}{\delta A_\alpha(x)}, \tag{22}$$

where S_A is defined as $Z[A] = \exp(iS_A)$.

In mean field theory we can replace S_A by S, as computed using the classical path, and equate fields with their expectation values. Using the static field equation $\frac{\delta S}{\delta a_\alpha} = 0$ we can write

$$j_\alpha = \frac{\delta S}{\delta A_\alpha} = \frac{\delta S_\phi}{\delta a_\alpha} = -\frac{\delta S_a}{\delta a_\alpha}. \tag{23}$$

After an integration by parts the Chern-Simon Lagrangian can be written in the form

$$\mathcal{L}_a = \frac{e\pi}{2\theta\phi_0}\epsilon^{\alpha\beta}(2a_\alpha\partial_\beta a_0 - a_\alpha\partial_t a_\beta). \tag{24}$$

Correspondingly

$$\frac{\delta S_a}{\delta a_\alpha} = \frac{e\pi}{2\theta\phi_0}\epsilon^{\alpha\beta}(2\partial_\beta a_0 - \partial_t a_\beta), \tag{25}$$

evaluating the expression for static $\mathbf{a}$ we obtain our final result

$$j_\alpha = \frac{e^2}{h}\frac{\pi}{\theta}\epsilon^{\alpha\beta}E_\beta \tag{26}$$

from which we can read off $\sigma_{xx} = 0$ and $\sigma_{xy} = \frac{1}{2k+1}\frac{e^2}{h}$.

3.4 Vortices

Still in the realm of mean field theory we can look at the equations of motion for the fields and see that in addition to the uniform ground state there exist vortex solutions, i.e. solutions for which

$$\oint_\ell \nabla\theta d\ell = \pm 2\pi n, \tag{27}$$

for vortices of strength n. Here $\theta(\mathbf{r})$ is the phase of the complex field $\phi(\mathbf{r})$. Far from a unit vortex located at the origin the solution for $\mathbf{a}$ takes the asymptotic form

$$\delta\mathbf{a} \equiv \mathbf{a}(\mathbf{r}) + \mathbf{A}(\mathbf{r}) = \pm\frac{\phi_0}{2\pi}\frac{1}{r}\hat{\mathbf{e}}_\phi. \tag{28}$$

Therefore for a large contour we have

$$\oint \delta\mathbf{a}\cdot d\ell = \pm\phi_0. \tag{29}$$

Because of (14)

$$\rho = \bar{\rho} + \delta\rho = \frac{\nu}{\phi_0}\epsilon^{\alpha\beta}\partial_\alpha a_\beta = \frac{\nu}{\phi_0}\epsilon^{\alpha\beta}\partial_\alpha(\delta a_\beta - A_\beta) = \frac{\nu}{\phi_0}\epsilon^{\alpha\beta}\partial_\alpha \delta a_\beta + \frac{\nu}{\phi_0}B. \tag{30}$$

Therefore

$$\delta\rho = \frac{\nu}{\phi_0}\epsilon^{\alpha\beta}\partial_\alpha\delta a_\beta, \tag{31}$$

which upon integration gives the excess charge of the vortex

$$Q = e\int d^2r\delta\rho(r) = e\frac{\nu}{\phi_0}\oint \delta a\cdot d\ell = \pm e\nu \tag{32}$$

This demonstrates that at the fractional filling the vortex excitation have the corresponding fractional charge.

If we separate the topologically nontrivial part $\widetilde{\phi}(r)$ of $\phi(r)$ by writing

$$\phi(r) = \sqrt{\rho(r)}e^{\imath\theta(r)}\widetilde{\phi}(r) \tag{33}$$

one can construct an effective theory for $\widetilde{\phi}(r)$ by integrating out the other degrees of freedom. We can use this theory to study the vortex dynamics. The Chern-Simons field theory formulation allows one to study in details the FQHE in a standard field theoretical framework.

References

[1] S. Zhang, H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989)

[2] S. Zhang, *Int. J. Mod. Phys. B* **6**, 25-58 (1992)

[3] V. J. Goldman, I. Karakurt , Jun Liu and A. Zaslavsky , *cond-mat*/0010093