Phase Transitions

Homework Sheet 7

Due Noon Fri 18 April 2008, in the 563 box.

Question 6–1.

Consider a system described by the Landau free energy density

$$\mathcal{L} = \frac{1}{2}a\eta^2 + \frac{1}{4}b\eta^4 + \frac{1}{6}c\eta^6 - h\eta$$

where $c > 0$, and a and b are both linearly proportional to pressure P and temperature T near the point (T_c, P_c): $a = a_1 t + a_2 p$ and similarly for b, with $t = (T - T_c)/T_c$ and $p = (P - P_c)/P_c$. As P and T are varied both a and b can be made to vanish and change sign. Such a system exhibits a *tricritical point*. An example is a mixture of He^3 and He^4. In this question, we will calculate the phase diagram in the $a - b$ plane for $h = 0$, using Landau theory. First, we set $h = 0$.

(a) Consider the case $a < 0$. Find the extrema of the Landau free energy, and study their stability. Hence show that

$$< \eta >^2 \equiv \eta_s^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2c}.$$

(b) Now consider the case $a > 0$, $b > 0$. How does η_s behave in this region?

(c) Lastly, study carefully the case $b < 0$, $a > 0$. What happens here?

(d) Now sketch the phase diagram in the a-b plane, indicating the order of any phase transitions that you have found, and the positions of the phase boundaries. Sketch the form of the Landau free energy in each of the different regions of the phase diagram. The point $a = 0$, $b = 0$ is called the tricritical point. Can you suggest why? (Hint, think of $h \neq 0$.)

(e) Calculate the thermodynamic critical exponents, by approaching the tricritical point along the line $b = 0$. What do you expect ν to be?

(f) Show that for b small but positive, there is a cross-over from the tricritical behaviour which you have found to ordinary critical behaviour when $b^2 \sim -ac$.

Question 7–2.

This question concerns the use of finite size scaling to estimate critical exponents and transition temperatures from transfer matrix calculations in a strip. Consider the $d = 2$ Ising model on a square lattice. There are N rows parallel to the x axis and M rows parallel to the y axis. We will require that $N \to \infty$ whilst we will calculate the transfer matrix for $M = 1$ and $M = 2$. Periodic boundary conditions apply in both directions, so that our system has the topology of a torus. The Hamiltonian is

$$\mathcal{H} = K \sum_{n=1}^{N} \sum_{m=1}^{M} S_{mn}S_{m+1n} + S_{mn}S_{mn+1}.$$

In a previous exercise, you constructed the transfer matrix for this problem, and calculated the eigenvalues λ_1 and λ_2. The correlation length is given by

$$\xi^{-1} = \log \lambda_1/\lambda_2.$$

Use finite size scaling and the results from HW 3–2 to estimate the critical value of K and the exponent ν.

Question 7–3.

(a) Using any of the mean field techniques that you have seen, work out the mean field theory phase diagram in the (T, Δ) plane for the Blume-Capel model, defined by the Hamiltonian

$$H = -\sum_{<ij>} J S_i S_j + \Delta \sum_i S_i^2, \quad (J > 0, -\infty < \Delta < \infty)$$

where each spin can take the values $S_i = -1, 0, +1$. Your phase diagram should indicate the nature of the phases, and the order of the transitions between them. Make sure that you locate any points where the order of the transition changes.

(b) Suppose the interaction had not been nearest neighbour, but was instead

$$\sum_{ij} J(\mathbf{r}_i - \mathbf{r}_j) S_i S_j,$$

where $\mathbf{r}_i$ is the position in space of the ith spin. What will be the result of (a) this time? Your answer should involve the Fourier transform of the exchange interaction $\hat{J}(\mathbf{k}) \equiv \sum_{\mathbf{r}} J(\mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r})$.

(c) Check that your phase diagram is reasonable by finding a region where the behaviour of the Blume-Capel model reduces to the usual spin-1/2 Ising model.