

# The Random Field Ising Model at Zero Temperature

Nir Friedman

4/28/08

## **Abstract**

Barkhausen noise and hysteresis are explained in detail, and requires that must be met to model it successfully. The random field Ising model is constructed using physical ideas. The dynamics of the model are discussed, as well as qualitative features and the reasoning behind why a critical point is expected. Theoretical methods, such as the epsilon expansion as well as scaling techniques are briefly discussed, as is the computation treatment of the random field Ising model. The results of theory, numerical simulations, and experiments are all compared. The usage of random field Ising model for systems that display power law scaling beyond mere magnetic materials is briefly demonstrated with an example from earthquake dynamics. Future directions for study are suggested at the end.

# 1 Introduction

The Ising model is one of the most widely used basic models in statistical mechanics. By appropriate choices of Hamiltonian and correct interpretation of results, it can model not only magnetic spin systems but a range of other phenomena such as non-ideal gases. However, when we deal with the normal Ising model in equilibrium there are two serious theoretical considerations that it has a hard time coping with. First of all, all of the functions that determine physical quantities in the Ising model are smooth except at a finite number of points in phase space (which have the potential to be phase transitions). While this may seem reasonable at first, there is no reason why in principle a real system cannot respond to stimuli in a series of small jumps, i.e. show discontinuous behaviour everywhere. Nor is it necessarily true that in the thermodynamic limit this “crackling” noise will somehow become irrelevant. Second, hysteretical effects are almost absent in the standard Ising model in any meaningful way, occurring only in the special case of zero magnetic field and below the critical temperature. Hysteresis occurs when the state of a system depends not just on its current configuration of parameters, but also on its recent configurations or “history”. This implies that for some given configuration of parameters, there is more than one stable state.

It is worth noting that at some level both these deficiencies are related to thermal fluctuation. In the first case, crackling noise and non-analyticity in general is reduced by thermal fluctuation, which causes an element of randomness to appear in the behaviour of spins. The sharp, sudden response of individual or small groups of spins which would take place in a deterministic setting is eliminated in favour of the smooth change of the average over all of the spin states in the ensemble. Hence it is very rare on the whole to see non-analytic behaviour in the Ising model (rare here in the sense of occupying a very small volume in phase space).

In the second case, the likelihood of hysteresis is undermined by thermal fluctuations allowing the sampling of system states in a vicinity around the current state. Recall that hysteresis requires multiple stable states. Without thermal fluctuations, changes only occur if they are immediately favourable and hence a stable state must merely be locally stable. Thermal fluctuation however implies that any states that are not separated by a large energy barrier are likely to be sampled, and hence a stable state in most cases must not only be locally stable but also the globally preferred state. In practical terms, when the system behaviour is governed by minimizing some function (such as free energy) this is the difference between the system being stable at the global minimum as opposed to any local minimum. It is quite general to see a function have multiple local minima over a range of parameters, but quite rare for it to have multiple global minima.

So we can see that the Ising model in its usual form is not ideal for modeling these phenomena. Before we move on to construct a model, it is useful to examine a physical system that actually displays both phenomena of interest. Consider our model of the magnetic properties of iron. Magnetism in iron is typically modelled by domains: that is there are many microscopically small regions and within each region the magnetic field is aligned in some direction. However each domain points in a different direction under ordinary circumstances, hence giving a net magnetization of zero. If however an external magnetic field is applied, then the boundaries of the domain change so that a larger portion of the material is taking up by domains with some particular magnetic orientation: hence we have a net internal magnetic field generated by the material. When we apply and then remove a magnetic field however, the domain boundaries do not return to exactly what they were in the first place. Hence we have a hysteretical effect here, a dependence on the “history” of the external magnetic field. Furthermore, if we make a plot of applied field versus magnetization, the curve may look superficially smooth however when we zoom in close we will see tiny jumps and bursts. These tiny jumps are bursts are Barkhausen noise. We can see both hysteretical effects and Birkhausen noise in figure 1 [1].

At this point, we have physical motivation to try and gain insight into how these phenomena can be understood and modeled. However it also would appear that the standard Ising model is not a good starting point for such a model. Thus we have some good incentive to try and construct a slightly different model. As noted before in detail, thermal fluctuations have a significant negative impact on our ability to model the kinds of phenomena we are discussing. Thermal fluctuations however

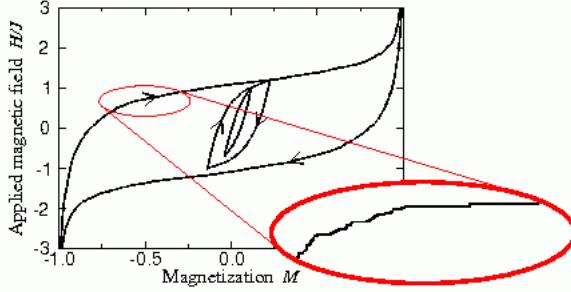


Figure 1: **Magnetization versus magnetic field** [1]. The sub-loops present in the figure demonstrate the presence of hysteresis. Notice how zero magnetization can correspond to many different levels of applied field. The zoomed inset shows the roughness of the curve on a small scale, implying that the magnetizing is changing in discrete jumps.

can of course be eliminated by setting the temperature equal to zero. However, a zero temperature Ising model is quite boring, because there is no source of randomness whatsoever. The spins will stay all up or all down, until the external field is brought past some threshold and then they will all flip simultaneously. If our intent was to functionally model a light switch (applied force analogous to magnetic field, light intensity analogous to magnetization) this might be effective, but is not very interesting behaviour. The physical origin of Barkhausen noise however gives us some insight on how to proceed. Recall that it is the flipping of magnetic domains that cause the noise, and that the reason that some flip before others is due to impurities, crystal dislocations, etc. In other words this is a result of random microscopic factors. To create a similar situation here, we would like to change for each spin site the propensity it has towards pointing one direction or the other [1]. In other words, we would like to add a local magnetic field. So, the Hamiltonian we have is

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i - \sum_i h_i S_i. \quad (1.1)$$

The interpretation of  $H$  and  $J$  are as usual. The role of  $h_i$  is to determine how easily the spin at site  $i$  is to flip relative to other spin sites. Given this purpose, there is no need for  $h_i$  to have a zero mode and hence we can expect that  $\sum_i h_i = 0$ . We should also expect that since the reference frame in which we choose indices is arbitrary, the manner in which the  $h_i$  are chosen should be translationally invariant. Based on how we are constructing our model, it makes sense that the  $h_i$  are randomized at the beginning of the simulation and are not subject to further change (i.e. they are a property of the material). Hence they would be determined by some selection on the space of multi-variate distributions.

## 2 The Model

The goal of this section will be to make the reader familiar with the mechanics of the model and some of its qualitative features. It is worthily discussing, at least briefly, the mechanics of this model at a fairly basic level since this is rather unfamiliar to a student of traditional statistical mechanics classes. There are no fluctuation here, no randomness (after the  $h_i$  are initialized), no partition function. Each and every individual spin's orientation is determined wholly deterministically based on the total local magnetic field, which is given by

$$H_i = H + J \sum_j S_j + h_i, \quad (2.1)$$

where the sum over  $j$  is taken only over the nearest neighbours of  $i$ . The spin at  $i$  will take the value given by the sign of  $H_i$  at the first possible opportunity. Most of the results in the next section are

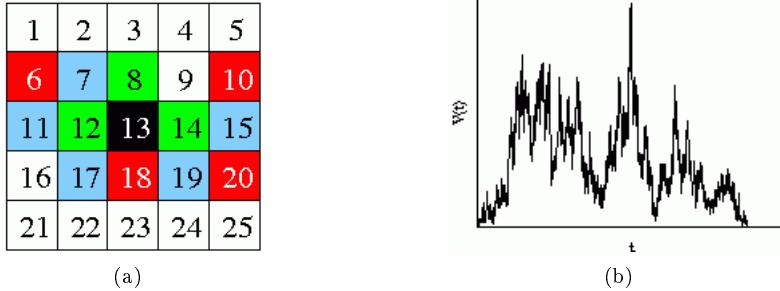


Figure 2: (a) **Schematic of an avalanche** [1]. This numbered grid represents a 2-D Ising model. The external magnetic field (which changes very slowly) caused the spin at 13 to flip. At this point, nearest-neighbour interactions caused a sequence of flips: first the spins located at the green boxes, then light blue, then red, at which point the sequence of flips or “avalanche” petered out. (b) **Time series of an avalanche** [1]. A typical avalanche time sequence,  $V$  is proportional to the number of flips at time  $t$ . The units of  $t$  are such that one increment of  $t$  is the time in which nearest-neighbour interactions can cause one flip. In the example from a), each colour represent flips that occurred simultaneously. Hence our time series would read  $V = \{1, 3, 5, 4\}$ .

dependent on three further assumptions. First of all, where dimension is not explicitly stated we take the dimension to be three. This is the natural choice to actually try and model a magnet. Second, when we collect data on the system,  $H$  will start at some value (typically very negative, so with certainty all the spins will be down) and sweep upwards very slowly. Very slowly means that at any point where the increase in  $H$  causes a spin to flip, we will first check if the changed nearest neighbour interaction causes spin flipping before we continue sweeping  $H$ . Thirdly, to proceed with our analysis we need to assume something about the distribution of the  $h_i$ . We will make the less realistic but more tractable assumption of taking the  $h_i$  to be samples on independent identically distributed Gaussian distributions of mean zero. Since they are independent, this implies that the correlation function,  $\langle h_i h_j \rangle = 0$ . This is clearly not very realistic, but the relative simplicity of analysis makes it a good starting point. The impact of this approximation and the possibility of not using will be discussed in more detail in the conclusion.

The fundamental reason why this model works well for modeling the Barkhausen noise discussed earlier is because a small change in  $H$  can cause a single spin to flip, which due to the nearest neighbour interaction causes many more spin flips without additional change in  $H$ . This event is referred to as an avalanche. These avalanches contribute substantially to crackling noise (since they have the potential to affect the magnetization much more suddenly than single spin flips). Figures 2(a) and 2(b) show how an avalanche develops spatially (in terms of the area around the initial spin flip) and temporally (in terms of how many spin flips occur at each instant after the initial flip).

Since we are collecting data by sweeping  $H$ , there is really only one other parameter in the system: the standard deviation of the Gaussian distributions, denoted by  $R$ . The behaviour with respect to this variable is critical then, because if a phase transition occurs, it must occur with respect to this variable. To see if something fundamentally changes with  $R$ , we can examine the limiting behaviour of the system. Suppose that  $R$  is zero. Then all the  $h_i$  are zero. In this case, we know exactly what will happen:  $H$  will rise up steadily from some large negative number, and nothing will happen, and  $H$  will become positive and still nothing will happen. At some point however  $H$  will become large enough to drown out the nearest neighbor tendency, but this will happen simultaneously for all spins. So we will have a single infinite avalanche. If on the other hand  $R$  becomes extremely large (relative to  $J$ ), then this is essentially the same as taking  $J$  very small. If there is no nearest neighbor interaction, then there will be no avalanches at all. The question now is whether there is some clear cut dividing line between these two behaviors, and the answer is yes. If  $R$  is below some value (call it  $R_c$ ) than at

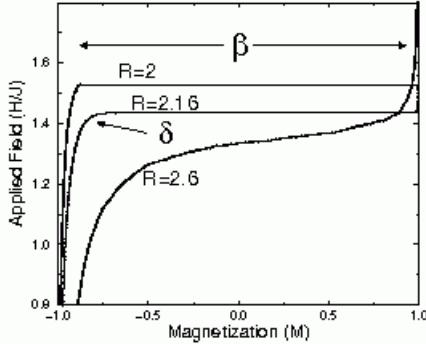


Figure 3: **Finite and infinite avalanches** [2]. We have  $R_c \approx 2.16$ . Hence we see that for  $R = 2 < R_c$ , we have a sudden discontinuity in the magnetization caused by a very large avalanche. For  $R = 2.6 > R_c$ , the curve is smooth (unless we zoom in very close). Notice also the appearance of critical exponents on this graph:  $\beta$  relates the size of the discontinuity in  $M$  to  $-(R - R_c)$ , and  $\delta$  relates  $(R - R_c)$  to  $(H - H_c)$ ,  $H_c$  being defined as the place where the largest avalanches occur.

some point there will be a single avalanche that has a finite impact even in the thermodynamic limit  $N \rightarrow \infty$ . In other words, an infinite number of spins flip simultaneously (although not necessarily every spin). If  $R$  is above this value, than all avalanches are finite. We will now take a look at some of the analysis performed to quantify this behavior.

### 3 Analysis

There are two primary methods of analysis here: theoretical, and numerical. The former is done via the renormalization group, and the latter is done via computer simulations. We do expect power law scaling near the critical point, as near  $R_c$ , the avalanches come in a distribution of all sizes. As commented by Sethna et al, at the boundary between huge avalanches and small ones, the system cannot make up its mind and so avalanches come in all sizes [5]. Although a full review of the mathematical techniques used is beyond the scope of this system, a brief description of the basic approach will be given.

The usual approach to finite temperature, equilibrium statistical mechanics is to write the partition function and so on. The situation here is a bit different. Once we have settled sampled our  $h_i$  for their Gaussian distribution, the system proceeds deterministically. In order to understand the behaviour of the system given a certain distribution however, we do indeed need to proceed by treating the  $h_i$  as just that. Hence instead of thinking moving along a deterministic path for some set of numbers  $h_i$ , we think of moving along the distribution of paths for some set of distributions  $h_i$ . Then we can calculate quantities by averaging over all the possible paths. Now, the paths themselves occur as a result of changing the external field very slowly, as a function of time. Hence we want to introduce explicit time dependence to our variables and write out equations which will govern how the system evolves. The first step is to represent  $H$  as

$$H(t) = H_0 + \Omega t, \quad (3.1)$$

where our goal is to take  $\Omega$  to zero at the end of the calculation (making the system purely adiabatic). We also now make the change from a discrete spin system to a continuous one. Once this is done we can more easily write dynamics for the system. To ensure however that our spins still settle on a

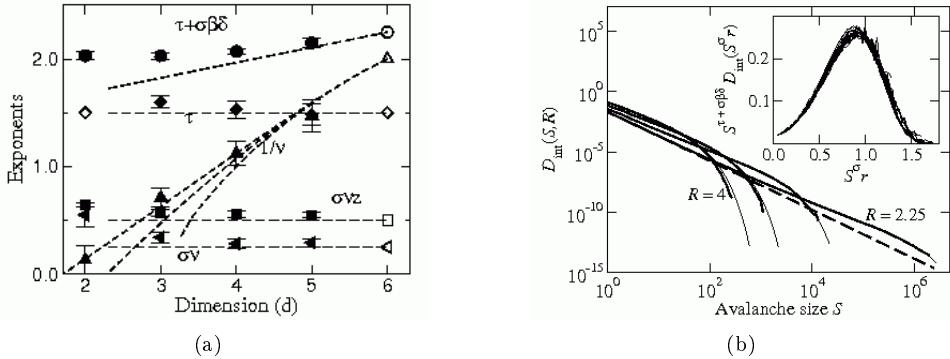


Figure 4: (a) **Theoretical versus numerical critical exponents** [2]. We can see in this diagram the surprising accuracy of the epsilon expansion for various critical exponents. (b) **Scaling laws** [1]. Although perfect power laws only occur at  $R = R_c$ , we can see that even for  $R = 4$ , we have two decades of power law scaling, which indicates that the critical region is large. The inset clearly shows data collapse. We can see from the form of the function that at criticality,  $r = 0$  and thus  $D_{int} \sim S^{-(\tau + \sigma\beta\delta)}$ .

value of  $\pm 1$ , we add a potential term to the Hamiltonian:

$$\mathcal{H} = - \sum_{ij} J_{ij} s_i s_j - \sum_i (h_i s_i + H s_i - V(s_i)), \quad (3.2)$$

where

$$V(s_i) = \begin{cases} k/2(s_i + 1)^2, & s < 0 \\ k/2(s_i - 1)^2, & s > 0 \end{cases}. \quad (3.3)$$

The Hamiltonian is the same as before, except that now the  $s_i$  are continuous variables that range from minus infinity to plus infinity. However the new potential term ensures that their values are localized around  $\pm 1$  as desired [6].

We can now write the equation governing the dynamics of the system as

$$\frac{1}{\Gamma_0} \frac{\partial s_i(t)}{\partial t} = - \frac{\delta \mathcal{H}}{\delta s_i(t)}, \quad (3.4)$$

where  $\Gamma_0$  is a frictional constant, i.e. it determines how quickly the dynamic variables change in response to stimuli. This leads to a formalism for the partition function:

$$\mathcal{Z} = \int [ds] \prod_i \delta(\partial_t s_i / \Gamma_0 + \delta \mathcal{H} / \delta s_i). \quad (3.5)$$

The idea is to integrate over all possible paths, and to keep only the ones which satisfy the equations of motion listed above. Eventually, the idea is to expand upon this by than integrating over the distribution of the  $h_i$  that appear in 3.5 through  $\mathcal{H}$ . The math, as discussed is quite long and difficult, and includes several techniques including introducing un-physical fields (or “ghosts” as they would often be known in field theories), transforming the spins to local fields via the transformation  $J\eta_i = \sum_{ij} J_{ij} s_j$ , and using mean field theory. A key point in this calculation is that in the mean field theory, the upper critical dimension is found to be six. Hence, any calculation of critical exponents for less than six dimensions occurs via potentially dubious asymptotic expansions in  $\epsilon$  (where we are working in  $6 - \epsilon$  dimensions). However, the results show surprisingly good agreement with simulation, as shown in figure 4 [6].

Much simpler in scope, if we take self-similarity at the critical point as a given we can make simple scaling arguments that allows to derive power law relations between variables. Consider  $L$ , the length scale we are coarse graining,  $S$ , the size scale of avalanches, and  $D(S)$ , the probability of finding some particular avalanche at a given size. We can derive the form of the scaling that appears in figure 4(b). We begin by assuming power law relations at criticality, then under some small coarsening defined by  $\epsilon$  we have

$$\begin{aligned} L' &= L/B = L/(1+\epsilon) \\ S' &= S/C = S/(1+c\epsilon) \\ D' &= A/D = D/(1+a\epsilon). \end{aligned} \quad (3.6)$$

Here,  $c$  is the fractal dimension, typically denoted by  $1/\sigma\nu$ . The ratio  $a/c$  is equal to  $\tau$ . We can use these equations to derive the basic power law behavior as a function of one variable. The likelihood of the new system having an avalanche of size  $S$  is the same (after rescaling  $D$ ) as the old system having an avalanche of size  $CS$ , hence we know that  $D'(S) = AD(CS)$ . But, we are also assuming that at criticality the system looks the same at all length scales. Hence to first order in  $\epsilon$  we have

$$\begin{aligned} D(S) &= D'(S) \\ &= (1+a\epsilon)D((1+c\epsilon S)) \\ 0 &= a\epsilon D + c\epsilon S \cdot \partial_S D \\ \frac{dD}{dS} &= -\frac{a}{c} \frac{D}{S}. \end{aligned}$$

This is satisfied if  $D \sim S^{-a/c} = S^{-\tau}$ . Hence  $\tau$  is our critical exponent relating avalanche size to likelihood at criticality. With a little more effort, we can also derive the two variable scaling form of  $D$  in terms of  $S$  and  $R$ , the standard deviation of the Gaussian distribution. If we work through the algebra we find

$$D(S, R) \sim S^{-\bar{\tau}} F_D(S^\sigma(R - R_c)), \quad (3.7)$$

where  $\bar{\tau}$  is a new critical exponent related to the hysteresis loop and satisfies  $\bar{\tau} = \tau + \sigma\beta\delta$ , and  $\sigma$  relates the disorder and the size-cutoff (i.e. the maximum size of avalanche we can expect to see) as  $S \sim (R - R_c)^{-\sigma}$ . This data collapse is precisely demonstrated by the inset to figure 4(b) [1, 2].

At the beginning of the section, the use of numerical methods was briefly discussed. This will not be discussed in great detail here as these are generally more to do with computation than with physics. However it is worth briefly mentioning the difference conceptually between the brute force algorithm and a slightly more sophisticated one. A brute force algorithm is fairly straightforward: there is an overall loop which increments  $H$  by some small amount, and each time this small increment is done each spin is checked to see if it is ready to flip. When a spin is flipped, the effects are propagated, and when the avalanche peters out the checking continues. This is very bad because the increments of  $H$  must be very small to match our analysis where we assumed the system was adiabatic, and if the increments are very small than for many of the increments not a single spin will change. Hence there is a huge amount of wasted effort. A much more efficient strategy is to store the local magnetic fields in an ordered list from the very beginning. When this is done, rather than increment  $H$  slowly we simply directly increase  $H$  by the smallest amount possible so as to cause one spin to flip. We propagate the effect of that spin both on other spin flips as well as in terms of how it affects the local magnetic fields, i.e. we make sure that at the end of the avalanche we still have a sorted list of local magnetic fields. We then simply iterate this process. By doing it this way instead, we raise  $H$  more rapidly, but by exactly the right amount so as to keep the simulation entirely adiabatic [3].

## 4 Results and Applications

The original motivation for this model came from Barkhausen noise and hysteresis in magnetic materials, so it makes sense to return to these cases (where there is an ample experimental trail) to see

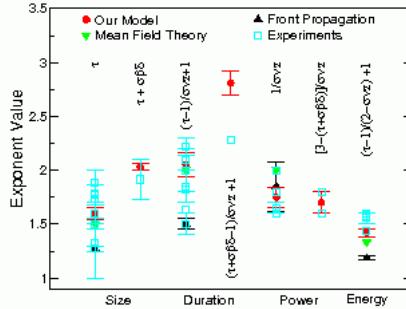


Figure 5: **Comparison of various models to experiment [2].** This graph compares three models to experiment. The front propagation is a model whereby domains are established at the beginning and flips are only allowed along the existing boundary (i.e. nucleation, the flip of a spin surrounded by unflipped spins, is not permitted). The “mean field theory” refers to a dipolar mean field theory that has differing interaction terms in its Hamiltonian. The red circles are the model of interest. We can see that all three models produce quite good results.

what kind of results the random field Ising model predicts, and how they compare to reality.

One of the most important results of this model is the power law relation between avalanche size and frequency. In many natural phenomena it is quite common for “larger” events to be rarer, and in many cases this is governed precisely by a power law. The fact that this model generates this relation and self-similarity so naturally and over a broad range make it a potential candidate for modeling such systems. As a particular example, consider the scaling effect in earthquake dynamics. The Gutenberg-Richter scaling law says that earthquake frequency goes as a power law of earthquake moment. This is very similar to our result with regards to avalanches. Note only that, but if we regard an earthquake as an accumulation of smaller faults all “slipping” at the same time, there is very good reason to believe that the random field Ising model is a logical beginning for understanding the origins because the Gutenberg-Richter scaling law. Recently Fisher et al. have had success modeling earthquake dynamics with equations of motion given by

$$\eta \partial_t u(\mathbf{r}, t) = F + \sigma(\mathbf{r}, t) - f_R[u(\mathbf{r}, t), \mathbf{r}, \{u(\mathbf{r}, t' < t)\}], \quad (4.1)$$

with  $\sigma$  given by

$$\sigma(\mathbf{r}, t) = \int_{-\infty}^t dt' \int d^d r' J(\mathbf{r} - \mathbf{r}', t - t')[u(\mathbf{r}', t') - u(\mathbf{r}, t)]. \quad (4.2)$$

Of course, this equations of motion are extremely similar to the ones we used to perform the epsilon expansion. The right hand side of 4.1 is nothing but the local magnetic field, and the right hand side the corresponding rate of change in the spin variable.  $\sigma$  is easily recognizable as the nearest neighbor interaction [4].

## 5 Conclusions and Future Directions

We can see that the random field Ising model has provided a powerful model of some important phenomena. Hopefully the potential of this and related models is only beginning to be untapped. There are many aspects of the random field Ising model that can be varied, such as the microscopic states available, the interactions, perhaps entering a finite temperature, and so on. Perhaps by tuning these it might be possible to find new universality classes that better model some of the particular systems we are interested in. Out of the factors that are readily changeable however, one that stands out is the choice of distribution for the random part of the local field,  $h_i$ . Recall that

for simplicity we chose these to be independent samples on Gaussian distributions. However, if we stand by our claim that these are suppose to represent material defects or imperfections, they would clearly be spatially correlated. While the theoretical analysis might be substantially complicated by the addition of correlations in the random local field, it would have very little impact on the computational simulations. All that is required is an additional part to the program to generate the  $h_i$  to have the desired correlation. It will be interesting to see what, if any impact this change would have on the universality class. If the system is truly invariant under coarse-graining at criticality, then coarse-graining the spins would cause the correlation between the sites to decrease until it would eventually vanish. However, this sort of argument is specious without concrete theory or simulation behind it. Hence we can be rest assured there is plenty of interesting work left to be done in the random field Ising model. [2]

## References

- [1] James P. Sethna, “Crackling Noise and Avalanches: Scaling, critical phenomena, and the renormalization group”. arXiv:cond-mat/0612418v1.
- [2] James P. Sethna, Karin Dahmen, and Olga Perkovic. “Random-Field Ising Models of Hysteresis”. arXiv:cond-mat/0406320v3
- [3] Matthew C. Kuntz, Olga Perkovic, Karin A. Dahmen, Bruce W. Roberts, and James P. Sethna, 1999, *Computing in Science and Engineering*, vol. **01**, no. 4, pp. 73-81.
- [4] Daniel S. Fisher, Karin Dahmen, Sharad Ramanathan, and Yehuda Ben-Zion. *Phys. Rev. Lett.* **78** 4885.
- [5] James P. Sethna, Olga Perkovic, and Karin A. Dahmen, March 10-14, 1997. Les Houches Workshop on “Scaling and Beyond”, p. 87.
- [6] Karin Dahmen and James P. Sethna, 1996, *Phys. Rev. B* **53** 14872.
- [7] Olga Perkovic, Karin Dahmen, and James P. Sethna, 1995, *Phys. Rev. Lett.* **75** 4528.